Workbook Unit 16:

Categorical Propositions

Overview

In this unit, we will continue to learn the basics of symbolization in predicate logic. We will introduce the so-called categorical propositions of classical logic and their symbolizations in predicate logic.

This unit

- teaches you to symbolize categorical propositions
- teaches you in what sense predicate logic is an extension of propositional logic

Prerequisites

You must have completed Units 13 and 14.

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1. Four Categorical Propositions

Long before predicate logic was developed by Gottlob Frege (in 19^{th} century), philosophers and logicians have worked out the so-called classical (or syllogistic) logic. Classical logic was began by Aristotle and developed particularly prominently by Medieval philosophers and theologians. Classical logic is certainly not as powerful as predicate logic, though some useful methods have been developed in it. What is interesting is that classical logicians distinguished four types of propositions (which they called categorical propositions), which have turned out to be very important in that they are really quite crucial in learning how to symbolize most English sentences into predicate logic.

Here are the four types of propositions: A, I, E and O.

A:	**All** men are jealous	**All** Ss are P
I:	**Some** women are ambitious	**Some** Ss are P
E:	**No** women are obnoxious	**No** Ss are P
O:	**Some** men are **not** crazy	**Some** Ss are **not** P

You will need to remember which is which, but the Medieval monks have already provided a useful mnemonic. The first two propositions A and I are affirmative, while the last two are negative. The names of the propositions come from the Latin for 'I affirm':

a*ff****i****rmo*

The universal affirmative is the A proposition, while the particular affirmative is the I proposition. The Latin for 'I deny' is:

*n****e****g****o***

The universal negative is the E proposition, while the particular negative is the O proposition.

Exercise "Categorical Propositions - 1"

(a)	All politicians are nasty.	Proposition:	
(b)	Some Democrats are not happy after the elections.	Proposition:	
(c)	Some Republicans are happy after the elections.	Proposition:	
(d)	No California Democrats are happy.	Proposition:	
(e)	Some American have not voted in the elections.	Proposition:	
(f)	Some voters were late to the booths.	Proposition:	
(g)	All Americans wanted to vote in the elections.	Proposition:	
(h)	No Americans need to make excuses.	Proposition:	

2. Categorical Propositions in Predicate Logic

Now that you can recognize categorical propositions, let us now consider how to symbolize them in predicate logic. Let us begin with the I and O propositions. Let us use the following symbolization key:

U.D.: people
Ax: *x* is ambitious
Cx: *x* is crazy.
Jx: *x* is jealous
Mx: *x* is a man
Ox: *x* is obnoxious
Wx: *x* is a woman

2.1. Proposition I: Some Ss are P

(I) Some women are ambitious

This proposition says that there is some woman who is ambitious, in other words, it says that there is someone in the U.D. who is both a woman and ambitious:

{I} ***There is some*** *x* such that *x* is a woman ***and*** *x* is ambitious.

In symbols:

[I] $\exists x\ (Wx \bullet Ax)$

2.2. Proposition O: Some Ss are not P

(O) Some men are not crazy

This proposition says that there is some man who is not crazy, in other words, it says that there is someone in the U.D. who is both a man and who is not crazy:

{O} ***There is some*** *x* such that *x* is a man ***and*** *x* is ***not*** crazy.

In symbols:

[O] $\exists x\ (Mx \bullet \sim Cx)$

2.3. Proposition A: All Ss are P

We are now turning to the universal propositions, starting with A:

(A) All men are jealous

You might think that the proposition would be symbolized analogically as:

$\forall x\ (Mx \bullet Jx)$

A little reflection and reading the above proposition will show that this is a mistake! For proposition $\forall x\ (Mx \bullet Jx)$ put in its canonical form reads:

For every *x*, *x* is a man and *x* is jealous.

In English: Everything is a jealous man. But this is not what we wanted to say!

Let us try again. What we want to say is that every man is jealous. In other words, for every member of the U.D., it will be jealous provided that it is a man, i.e.:

{A} ***For every*** *x*, ***if*** *x* is a man ***then*** *x* is jealous.

Now we have managed to say what we wanted: all men are jealous

[A] $\forall x\ (Mx \rightarrow Jx)$

2.4. Proposition E: No Ss are P

We will proceed analogically for the universal negative proposition E:

(E) No women are obnoxious

What we want to say is that no woman is obnoxious. In other words, for every member of the U.D., if it is a woman then it will not be obnoxious, i.e.:

{E} ***For every*** *x*, ***if*** *x* is a woman ***then*** *x* is ***not*** obnoxious.

Now we have managed to say what we wanted: no women are obnoxious

[E] $\forall x\ (Wx \rightarrow {\sim}Ox)$

All Ss are P	A	$\forall x\ (Sx \rightarrow Px)$
No Ss are P	E	$\forall x\ (Sx \rightarrow {\sim}Px)$
Some Ss are P	I	$\exists x\ (Sx \bullet Px)$
Some Ss are **not** P	O	$\exists x\ (Sx \bullet {\sim}Px)$

Exercise "Categorical Propositions - 2"

Symbolize the following opinions about politicians using the symbolization key provided. For each of the propositions, write down the canonical interpretation.

U.D.: politicians	Ax: x is ambitious	Ix: x is intelligent
	Cx: x is corrupt	Nx: x is new to politics
	Dx: x is dilligent	Px: x is pretentious
	Hx: x is honest	Tx: x is tired

(a) Some intelligent politicians are corrupt.

(b) There is an intelligent politician who is honest.

(c) Some corrupt politicians are intelligent.

(d) Some corrupt politicians are not intelligent.

(e) Some ambitious politicians are not honest.

(f) All corrupt politicians are ambitious.

(g) Any politician who is new to politics is honest.

(h) No corrupt politicians are honest.

(i) No honest politician is corrupt.

(j) All honest politicians are tired.

(k) No politician who is new to politics is tired.

(l) No honest politician is pretentious.

Exercise “Categorical Propositions - 3”

Symbolize the following propositions. For each of the propositions, write down the canonical interpretation.

U.D.: animals	Bx: x barks	Hx: x howls
	Cx: x is a cat	Lx: x likes to walk
	Dx: x is a dog	Mx: x meows
	Fx: x likes canned food	Wx: x wags its tail

(a) Some dogs howl.

(b) No cats howl.

(c) Some animals howl.

(d) Some cats do not like canned food.

(e) All cats meow.

(f) No cat likes to walk.

(g) All dogs wag their tails.

(h) All animals like to walk.

3. Quantifier Scope, Free and Bound Variables

Just as in propositional logic so in predicate logic, you need to be able to tell what the main operator of a proposition is. The introduction of quantifiers complicates matters somewhat. The scope of the quantifiers is analogical to the scope of negations – they apply to what immediately follows it. Consider the following externally simple quantified propositions

(1) $\forall x\ Nx$
(2) $\forall x\ (Px \vee Nx)$
(3) $\exists x\ {\sim}((Px \vee Nx) \vee (Cx \bullet {\sim}Px))$

In all of these propositions the quantifier "reaches" the end of the formula, as marked below:

(1) $\forall x\ \boxed{Nx}$
(2) $\forall x\ \boxed{(Px \vee Nx)}$
(3) $\exists x\ \boxed{{\sim}((Px \vee Nx) \vee (Cx \bullet {\sim}Px))}$

In propositions (2)-(3), the reason why the quantifier's scope "reaches" the end of the formula is guaranteed by the parentheses. Were the parentheses not to be there, the quantifier would reach only as far the first propositional function; consider the following formulas:

(4) $\forall x\ Px \vee Nx$
(5) $\exists x\ {\sim}(Px \vee Nx) \vee (Cx \bullet {\sim}Px)$

Can you draw how far the quantifier scope extends? (Peak on the next page, to check that you have done it correctly.) The variables that fall within the scope of the quantifier are **bound** (by the quantifier), while the variables that fall outside the scope of the quantifier are **free**.

The distinction between free and bound variables is very important because formulas that include at least one free variable are not propositions but propositional functions.

> If a formula includes at least one free variable then it is a propositional function. Propositions include only bound variables.

The reason why a formula that includes a free variable is a propositional function can be understood if you contrast the following formulas (let U.D. be people; *Wx* – *x* is a woman; *Gx* – *x* is gorgeous):

(6)	$\forall x\ Wx$	Everybody is a woman.
(7)	$\forall x\ (Wx \rightarrow Gx)$	Every woman is gorgeous.
(8)	$\forall x\ Wx \rightarrow Gx$	If everybody is a woman then *x* is gorgeous.

Formulas (6) and (7) are propositions, while (8) is a propositional function because the last occurrence of variable *x* is not bound by any quantifier.

Here is a list of all the formulas discussed in this section without and with the quantifier scope marking.

(1)	$\forall x\ Nx$	(1)	$\forall x\ Nx$
(2)	$\forall x\ (Px \vee Nx)$	(2)	$\forall x\ (Px \vee Nx)$
(3)	$\exists x \sim((Px \vee Nx) \vee (Cx \bullet \sim Px))$	(3)	$\exists x \sim((Px \vee Nx) \vee (Cx \bullet \sim Px))$
(4)	$\forall x\ Px \vee Nx$	(4)	$\forall x\ Px \vee Nx$
(5)	$\exists x \sim(Px \vee Nx) \vee (Cx \bullet \sim Px)$	(5)	$\exists x \sim(Px \vee Nx) \vee (Cx \bullet \sim Px)$
(6)	$\forall x\ Wx$	(6)	$\forall x\ Wx$
(7)	$\forall x\ (Wx \rightarrow Gx)$	(7)	$\forall x\ (Wx \rightarrow Gx)$
(8)	$\forall x\ Wx \rightarrow Gx$	(8)	$\forall x\ Wx \rightarrow Gx$

Exercise "Free and Bound Variables"

Show which variables are free and determine whether the formula is a proposition or a propositional function.

(a) $\forall x\ Px$ ❑proposition ❑propositional function

(b) $\forall x\ (Px \bullet Qx)$ ❑proposition ❑propositional function

(c) $\forall x\ (Px \bullet Qx) \rightarrow Rx$ ❑proposition ❑propositional function

(d) $\forall x\ Px \bullet Qx$ ❑proposition ❑propositional function

(e) $\exists x\ Px \equiv Qx$ ❑proposition ❑propositional function

(f) $\exists x \sim Px \bullet Qx$ ❑proposition ❑propositional function

(g) $\exists x\ (\sim Px \bullet Qx)$ ❑proposition ❑propositional function

(h) $\forall x\ (Px \bullet Qx) \rightarrow \sim(Px \bullet Rx)$ ❑proposition ❑propositional function

(i) $\exists x \sim(Px \bullet Qx)$ ❑proposition ❑propositional function

(j) $\exists x\ (\sim(Px \rightarrow Qx) \bullet \sim(Px \bullet Rx))$ ❑proposition ❑propositional function

(k) $\exists x\ (\sim(Px \rightarrow Qx) \bullet \sim(Px \bullet Rx)) \vee \sim(Rx \rightarrow Cx)$ ❑proposition ❑propositional function

4. Only Ss are P

[You should go back and remind yourself at this point how to symbolize 'only if' phrases, discussed in Unit 3.]

Consider the following true proposition:

(1) Only women are mothers

There are a number of equivalent ways of symbolizing (1) in predicate logic:

{1} For every *x*, if *x* is mother then [this means that] *x* is a woman.
[1] $\forall x\ (Mx \rightarrow Wx)$
{1′} For every *x*, if *x* is not a woman then *x* is not a mother.
[1′] $\forall x\ (\sim Wx \rightarrow \sim Mx)$

Consider another example.

(2) Only logicians do not have problems with the symbolization of 'only'.

Here are the equivalent ways of symbolizing (2):

{2} For every *x*, if *x* does not have problems with the symbolization of 'only' [this means that] *x* is a logician.
[2] $\forall x\ (\sim Px \rightarrow Lx)$
{2′} For every *x*, if *x* is not a logician then *x* has problems with the symbolization of 'only'.
[2′] $\forall x\ (\sim Lx \rightarrow \sim\sim Px)$
[2″] $\forall x\ (\sim Lx \rightarrow Px)$

All Ss are P	A	$\forall x\ (Sx \rightarrow Px)$
Some Ss are P	I	$\exists x\ (Sx \bullet Px)$
No Ss are P	E	$\forall x\ (Sx \rightarrow \sim Px)$
Some Ss are **not** P	O	$\exists x\ (Sx \bullet \sim Px)$
Only Ss are P		$\forall x\ (Px \rightarrow Sx)$

Exercise " 'Only' Propositions – 1"

Symbolize the following propositions using the symbolization key provided. For each of the propositions, write down the canonical interpretation.

U.D.: animals	*Bx*: *x* barks	*Hx*: *x* howls
	Cx: *x* is a cat	*Lx*: *x* likes to walk
	Dx: *x* is a dog	*Mx*: *x* meows
	Fx: *x* likes canned food	*Wx*: *x* wags its tail

(a) Only dogs bark.

(b) Only cats meow.

(c) Only dogs howl.

(d) Only dogs wag their tails

(e) Only dogs like to walk.

(f) Only cats like canned food.

(g) Only animals that bark like to walk.

(h) Only animals that like to walk wag their tails.

(i) Only cats do not like to walk.

(j) Only animals that meow like canned food.

Exercise " 'Only' Propositions – 2"

Symbolize the following propositions using the symbolization key provided. For each of the propositions, determine whether it is true or false. Some of the symbolizations will actually help you to reach a decision about the truth-values.

(For the purposes of the exercises, let us assume that women sometimes wear ties, but that men never wear skirts – the Scottish men wear kilts, but they insist that this is a different piece of clothing.)

U.D.: people	Fx: *x* is a father	Ox: *x* is a mother	Tx: *x* wears ties.
	Mx: *x* is a man	Sx: *x* wears skirts	Wx: *x* is a woman

(a)	All men are fathers.		❑ true ❑ false
(b)	Only men are fathers.		❑ true ❑ false
(c)	All women are mothers.		❑ true ❑ false
(d)	Only women are mothers		❑ true ❑ false
(e)	All mothers are women.		❑ true ❑ false
(f)	Only mothers are women.		❑ true ❑ false
(g)	Only men wear ties.		❑ true ❑ false
(h)	Only women wear skirts.		❑ true ❑ false
(i)	Only women do not wear ties.		❑ true ❑ false
(j)	Only men do not wear skirts.		❑ true ❑ false
(k)	Only persons wearing skirts are women		❑ true ❑ false
(l)	Only persons wearing ties are men.		❑ true ❑ false
(m)	Only persons wearing skirts do not wear ties.		❑ true ❑ false
(n)	Only persons who are not mothers are fathers.		❑ true ❑ false
(o)	Only persons who do not wear skirts are fathers.		❑ true ❑ false

5. Limitations of Propositional Logic

Recall what we said at the very beginning. Propositional logic is just not powerful enough to help us understand the validity of certain arguments. It allows us to understand why arguments such as (A_1) are valid but it does not help us understand why arguments such as (A_2) are valid.

(A_1)	If Socrates is human then he is mortal.	$H \rightarrow M$	
	Socrates is human.	H	
	So: Socrates is mortal.	M	A: All humans are mortal H: Socrates is human M: Socrates is mortal
(A_2)	All humans are mortal.	A	
	Socrates is human.	H	
	So: Socrates is mortal.	M	

We have now seen enough of predicate logic to see how predicate logic can help us in seeing the validity not only of (A_1) but also of (A_2).

The reason why predicate logic enables us to understand why (A_2) is also valid is that it enables us to reach deeper into the structure of the first premise of that argument "All humans are mortal", which, in predicate logic, is simply an A type proposition, i.e. an internally complex universal proposition. The arguments will be symbolized thus:

(A_1)	If Socrates is human then he is mortal.	$Ha \rightarrow Ma$	
	Socrates is human.	Ha	
	So: Socrates is mortal.	Ma	U.D.: people *a*: Socrates *Hx*: *x* is human *Mx*: *x* is mortal
(A_2)	All humans are mortal.	$\forall x\ (Hx \rightarrow Mx)$	
	Socrates is human.	Ha	
	So: Socrates is mortal.	Ma	

We have not studied the proof method in predicate logic (which is the only method for checking validity of arguments in predicate logic – there is nothing that corresponds to the truth table method), but you should see that the symbolization of (A_1) in predicate logic renders the argument simply an instance of M.P. It will be more difficult for you to see the pattern in (A_2) because seeing the pattern presupposes that you know another inference rule, called Universal Instantiantion, which allows us to derive an instance of a universal proposition for any member of the U.D. Since Socrates (*a*) is a member of the U.D., the rule of Universal Instantiation allows us to derive the proposition "If Socrates is human then he is mortal" ($Ha \rightarrow Ma$) from the proposition "All humans are mortal" ($\forall x\ (Hx \rightarrow Mx)$). The rest is just a matter of applying M.P. again.

The crucial point here is that the ability to recognize the validity of an argument such as (A_2) depends on the ability to analyze the structure of the argument properly. Indeed, it is because propositional logic did not have the tools to analyze the deeper logical structure of 'All Ss are P' propositions that it was unable to recognize the validity of (A_2)-type arguments. But the quest goes on. There are many arguments that we would consider to be valid that predicate logic cannot recognize as valid.

(B_1) All great cats like all antelopes.
All cheetahs are great cats.
All impalas are antelopes.
So: all cheetahs like all impalas.

To recognize the validity of (B_1) predicate logic needs to be extended to the logic of relations, which allows to symbolize (B_1) thus:

$$\begin{array}{l} \forall x\,(Gx \rightarrow \forall y\,(Ay \rightarrow Lxy)) \\ \forall x\,(Cx \rightarrow Gx) \\ \forall y\,(Iy \rightarrow Ay) \\ \hline \forall x\,(Cx \rightarrow \forall y\,(Iy \rightarrow Lxy)) \end{array}$$

You might or might not see the roots of validity of the argument thus symbolized. The important thing for you to have learned is that logic is by no means a closed science. The twentieth century in particular has seen a boom in the development of various kinds of logic:

- deontic logic (which analyzes the logical relations among propositions that include operators such as ‘has the right to’, ‘is obliged to’),
- modal logic (which analyzes the logical relations among propositions that include operators such as ‘possibly’, ‘necessarily’),
- temporal logic (which analyzes the logical relations among propositions that include operators such as ‘it has at some time been the case that’, ‘it will at some time be the case that’, ‘it has always been the case that’, ‘it will always be the case that’),
- relevance logic (which places relevance conditions on the conditionals),
- many-valued logic (which allows more than two truth-values),
- epistemic logic (which analyzes the logical relations among propositions that include operators such as ‘knows’, ‘believes’), and many others.

and the list is open-ended.

What You Need to Know and Do

- You need to be able to symbolize categorical propositions.
- You need to be able to symbolize “only” propositions.
- You need to be able to determine whether a formula includes any free variables and is thus a proposition or a propositional form.