Workbook Unit 15:

Quantifiers and Negation

Overview

In this unit, we will begin introducing internally complex quantified propositions. We will consider four possible ways in which the negation symbol and the quantifiers can occur. We will also learn two different equivalent ways of symbolizing the English expressions "Nobody" and "Not everybody".

This unit

- teaches you how to symbolize negation in quantifier expressions
- teaches you the Negated Quantifiers Equivalences

~∀*x Px* :: ∃*x* ~*Px*
~∃*x Px* :: ∀*x* ~*Px*

Prerequisites

You must have completed Unit 14.

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1. Quantifiers and Negation

There are exactly four possible ways in which the negation symbol and our two quantifiers can occur in a proposition:

[1] $\sim\exists x\, Bx$ [2] $\exists x \sim Bx$
[3] $\sim\forall x\, Bx$ [4] $\forall x \sim Bx$

We have already seen how to interpret propositions of the form [1] and [3] in the last unit. Propositions [1] and [3] are negations of quantified propositions. Propositions [2] and [4] are quantified negations. We will now consider all four more systematically. Let us adopt the following symbolization key:

U.D.: people
Bx: *x* is beautiful

The **negation of an existential proposition:**

[1] $\sim\exists x\, Bx$

is a negation of the proposition "There is a person who is beautiful", so we can read it:

It is not the case that: there is a person who is beautiful.
There is no person who is beautiful.

or more idiomatically:

(1) Nobody is beautiful.

To interpret the **existentialization of a negation**, on the other hand:

[2] $\exists x \sim Bx$

it will pay to read off what can be called "*the canonical reading*" of [2]:

{2} There is an *x* such that it is not the case that *x* is beautiful.

or, more perspicuously:

{2} There is an *x* such that *x* is not beautiful.

{2} simply says:

(2) Somebody is not beautiful.
(2) There is a person who is not beautiful.

The **negation of a universal proposition:**

[3] $\sim\forall x\ Bx$

is a negation of the proposition "Everybody is beautiful", so we can read it:

It is not the case that: everybody is beautiful.

or more idiomatically:

(3) Not everybody is beautiful.

To interpret the **universalization of a negation**, on the other hand:

[4] $\forall x \sim Bx$

it will pay to read off what the canonical reading of [4]:

{4} For every *x*, it is not the case that *x* is beautiful.

or

{4} For every *x*, *x* is not beautiful.

Think about this for a second. Let the following grid represent our universe of discourse – each cell represents one person:

1	2	3	4	5
6	7	8	9	10

Let us think about what {4} says. Consider person 1 in our simplified U.D. According to {4}, will this person be (a) beautiful, (b) not beautiful or (c) {4} does not specify? How about person 2? Will person 2 be (a) beautiful, (b) not beautiful or (c) {4} does not specify? Will person 3 be (a) beautiful, (b) not beautiful or (c) {4} does not specify? What about the others?

In each of these cases, you should have answered that according to {4}, the person will be not beautiful, because {4} says that *for every x, x is not beautiful!* In other words, our simplified U.D. might look as depicted in the *Solutions*. This means that {4} simply says:

Everybody is not beautiful.

or more idiomatically:

Nobody is beautiful.

You will be quite correct to remember that yet another formula can be used to symbolize the proposition "Nobody is beautiful", viz. $\sim\exists x\ Bx$. In fact, in predicate logic, propositions $\sim\exists x\ Bx$ and $\forall x \sim Bx$ are logically equivalent.

Exercise “Canonical Reading”

Using the different symbolization keys provided (1)-(5), provide both the canonical reading as well as the most idiomatic English reading of each of the formulas (a)-(d).

Note that in symbolization keys (3)-(5) the U.D. is different from the one we have discussed. Given the U.D. of dogs the quantifiers will be read as “All dogs” and “Some dog,” respectively. Given the U.D. of husbands the quantifiers will be read as “All husbands” and “Some husband,” respectively. Given the U.D. of things the quantifiers will be read as “Everything” and “Something,” respectively.

(1)	U.D.: people	Px: x is wise
(2)	U.D.: people	Px: x is happy
(3)	U.D.: dogs	Px: x barks
(4)	U.D.: husbands	Px: x cheats on his wife
(5)	U.D.: things	Px: x is round

(a) $\exists x \sim Px$

(1)

(2)

(3)

(4)

(5)

(b) $\sim\exists x\ Px$

(1)

(2)

(3)

(4)

(5)

(1)	U.D.: people	*Px*: *x* is wise
(2)	U.D.: people	*Px*: *x* is happy
(3)	U.D.: dogs	*Px*: *x* barks
(4)	U.D.: husbands	*Px*: *x* cheats on his wife
(5)	U.D.: things	*Px*: *x* is round

(c) $\sim\forall x\, Px$

(1)

(2)

(3)

(4)

(5)

(d) $\forall x\, \sim Px$

(1)

(2)

(3)

(4)

(5)

2. Negated Quantifier Equivalences

It turns out that the pairs of propositions linked by arrows in the following diagram are logically equivalent (we take the U.D. to be maximal, i.e. including everything, and we let '*Cx*' abbreviate '*x* is certain'):

3	**~∀*x* *Cx*** It is not the case that for every *x*, *x* is certain. Not everything is certain.	**~∃*x* *Cx*** It is not the case that there is an *x* such that *x* is certain. Nothing is certain.	**1**
4	**∀*x* ~*Cx*** For every *x*, *x* is not certain. Everything is uncertain.	**∃*x* ~*Cx*** There is an *x* such that *x* is not certain. Something is uncertain.	**2**

We will do two things. First, we will consider each of the pairs of propositions to see that their equivalence is indeed intuitive (§2.1-§2.2). Second, I will suggest to you a way to memorize the equivalences (§2.3). Ultimately, you will have to remember them to be able to use both in symbolizations.

2.1. ~∀*x* *Cx* is equivalent to ∃*x* ~*Cx*

Consider the existential proposition

[2] ∃*x* ~*Cx*

which given the above symbolization key, we have rendered as

(2) Something is uncertain.

Let us use a pictorial representation of our universe of discourse. This time each cell represents some thing. If a thing is certain we will use a ✓ symbol, if a thing is uncertain we will use a ✘ symbol.

Let us then represent what proposition (1) says. It says that there is at least one thing that is uncertain:

		✘		

Now to see that the proposition ~∀*x* *Cx* follows, let us ask the question: Given that (2) is true:

Is everything certain?

It seems clear that the answer must be negative – not everything is certain because there is at least one thing that is uncertain. The proposition that not everything is certain is expressed by the negation of the universal proposition:

[3] $\sim\forall x\ Cx$

which given the above symbolization key, we have rendered as

(3) Not everything is certain.

2.2. $\sim\exists x\ Cx$ is equivalent to $\forall x \sim Cx$

Consider the universal proposition

[4] $\forall x \sim Cx$

which given the above symbolization key, we have rendered as

(4) Everything is uncertain.

Let us again represent what proposition (4) says pictorially. It says that everything is uncertain, i.e.:

✘	✘	✘	✘	✘
✘	✘	✘	✘	✘

Now to see that the proposition $\sim\exists x\ Cx$ follows, let us ask the question: Given that (4) is true:

Is something certain?

It seems clear that the answer must be negative – nothing is certain because everything is uncertain. The proposition that nothing is certain is expressed by the negation of the existential proposition:

[1] $\sim\exists x\ Cx$

which given the above symbolization key, we have rendered as

(1) Nothing is certain.

2.3. How to Memorize the Equivalences?

Here are the equivalent propositions once again: [1] is equivalent to [4] and [3] is equivalent to [2]. After you understood why they are equivalent, it will pay to memorize. My teacher taught me a useful mnemonic device to help remember the equivalences, and I pass it on to you. Think about [1] as being transformed into [4]: the tilde that precedes the existential quantifier jumps over the quantifier changing it into the universal quantifier and seats itself in front of the propositional function. Now consider the process of transforming [3] into [2]: the tilde that precedes the universal quantifier jumps over the quantifier changing it into the existential quantifier and seats itself in front of the propositional function. (You can view the transformation process in the PowerPoint presentation.)

[1] $\sim\exists x\, Cx \quad :: \quad \forall x \sim Cx$ [4]

[3] $\sim\forall x\, Cx \quad :: \quad \exists x \sim Cx$ [2]

(The '::' is read "is logical equivalent to".)

Exercise "Negated Quantifiers"

Symbolize the following propositions in two equivalent ways:

U.D.: people *Hx*: *x* is honest *Tx*: *x* is trustworthy

(a) Not everybody is honest.

(b) Nobody is honest.

(c) Somebody is dishonest.

(d) Everybody is dishonest.

(e) Not everybody is trustworthy.

(f) Nobody is trustworthy.

(g) Somebody is untrustworthy.

(h) Everybody is untrustworthy.

(i) Not everybody is untrustworthy.

(j) Somebody is not dishonest.

What You Need to Know and Do

- You need to be able to symbolize negations of quantified propositions as well as quantifications of negations.

Further Reading

You can read about these matters further in a number of logic textbooks. I enclose the chapters titles for the textbooks I have chosen as optional.
Klenk: Ch. 11.3. Negated Quantifiers
Copi & Cohen: Ch.10.2. Quantification.
Hurley: Ch 8.1. Symbols and Translation