Solutions to Workbook Exercises

Unit 12:

Natural Deduction Proofs (III)

Ex. →Int.I.a.

		Pr.
3	B	Assp. (→Int)
8	A	
9	B → A	→ Int 3–8

		Pr.
3	~B	Assp. (→Int)
8	A	
9	~B → A	→ Int 3–8

		Pr.
3	C	Assp. (→Int)
8	~B	
9	C → ~B	→ Int 3–8

		Pr.
3	A ≡ C	Assp. (→Int)
8	B	
9	(A ≡ C) → B	→ Int 3–8

		Pr.
3	C	Assp. (→Int)
8	B → A	
9	C → (B → A)	→ Int 3–8

		Pr.
3	C → A	Assp. (→Int)
8	B	
9	(C → A) → B	→ Int 3–8

		Pr.
3	~(A → B)	Assp. (→Int)
8	C	
9	~(A → B) → C	→Int 3-8

		Pr.
3	~A	Assp. (→Int)
8	B → C	
9	~A → (B → C)	→ Int 3–8

		Pr.
3	A ∨ ~A	Assp. (→Int)
8	B	
9	(A ∨ ~A) → B	→ Int 3–8

		Pr.
3	B → C	Assp. (→Int)
8	C → D	
9	(B → C) → (C → D)	→ Int 3–8

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Ex. →Int.I.b. In all of the below proof schemas, you are asked to apply the →Int rule to derive a conditional (line 9). To do that we need construct a subderivation, which has already been constructed. Your task is to fill in the assumption of the subderivation (step 3) and the conclusion that would have be derived in step 8. (As before, the point of the exercise is not to actually construct the whole proof!)

		Pr.
3	A	Assp. (→Int)
8	B	
9	A → B	→ Int 3–8

		Pr.
3	~A	Assp. (→Int)
8	B	
9	~A → B	→ Int 3–8

		Pr.
3	A	Assp. (→Int)
8	~B	
9	A → ~B	→ Int 3–8

		Pr.
3	A ∨ C	Assp. (→Int)
8	B	
9	(A ∨ C) → B	→ Int 3–8

		Pr.
3	A	Assp. (→Int)
8	B → C	
9	A → (B → C)	→ Int 3–8

		Pr.
3	A → B	Assp. (→Int)
8	C	
9	(A → B) → C	→ Int 3–8

		Pr.
3	A ≡ ~B	Assp. (→Int)
8	~B ≡ A	
9	(A ≡ ~B) → (~B ≡ A)	→ Int 3–8

		Pr.
3	~(A → B)	Assp. (→Int)
8	~A → B	
9	~(A→B)→(~A→B)	→ Int 3–8

1.1. Proofs Using →Int

Example 2.

Prove that (A • C) → ~D follows from A → (B • ~D):

1.	A → (B • ~D)	Pr.
2.	| A • C	Assp. (→Int)
3.	| A	•Elim 2
4.	| B • ~D	→Elim 1, 3
5.	| ~D	•Elim 4
6.	(A • C) → ~D	→Int 2-5

Example 3.

Prove that from the premise A → (B → C) we can derive the conclusion (A • B) → C

1.	A → (B → C)	Pr.
2.	| A • B	Assp. (→Int)
3.	| A	•Elim 2
4.	| B → C	→Elim 1, 3
5.	| B	•Elim 2
6.	| C	→Elim 4, 5
7.	(A • B) → C	→Int 2-6

Exercises on Proofs with →Int

→Int.II. Complete the following proofs. First, you have to complete the preparation of the subderivation filling in the assumption in line 3 and the wanted conclusion in line 5. You should then fill in the missing step in line 4 and justify all steps:

1.	$A \rightarrow (B \vee C)$	Pr.
2.	$\sim B$	Pr.
3.	A	Assp. (→Int)
4.	$B \vee C$	→Elim 1,3
5.	C	DS 2,4
6.	$A \rightarrow C$	→ Int 3–5

1.	$A \rightarrow (B \equiv \sim D)$	Pr.
2.	$\sim D$	Pr.
3.	A	Assp. (→Int)
4.	$B \equiv \sim D$	→Elim 1,3
5.	B	≡Elim 2,4
6.	$A \rightarrow B$	→ Int 3–5

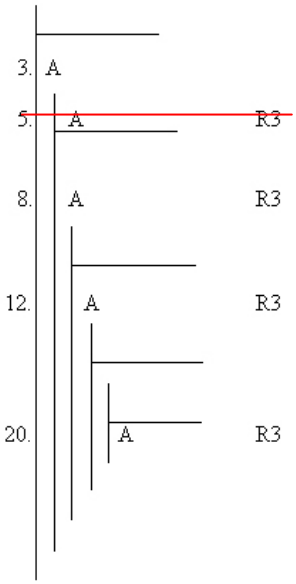
1.	$\sim A \rightarrow (C \rightarrow B)$	Pr.
2.	C	Pr.
3.	$\sim A$	Assp. (→Int)
4.	$C \rightarrow B$	→Elim 1,3
5.	B	→Elim 2,4
6.	$\sim A \rightarrow B$	→ Int 3–5

1.	$B \vee (B \vee C)$	Pr.
2.	$\sim B$	Pr.
3.	A	Assp. (→Int)
4.	$B \vee C$	DS 1,2
5.	C	DS 2,4
6.	$A \rightarrow C$	→ Int 3–5

1.	$A \equiv (B \vee C)$	Pr.
2.	$\sim D$	Pr.
3.	$A \equiv \sim D$	Assp. (→Int)
4.	A	≡Elim 3, 2
5.	$B \vee C$	≡Elim 1, 4
6.	$(A \equiv \sim D) \rightarrow (B \vee C)$	→ Int 3–5

1.	$(A \equiv B) \vee (B \vee C)$	Pr.
2.	$\sim(A \equiv B)$	Pr.
3.	$\sim B$	Assp. (→Int)
4.	$B \vee C$	DS 1,2
5.	C	DS 3,4
6.	$\sim B \rightarrow C$	→ Int 3–5

(a)



(b)

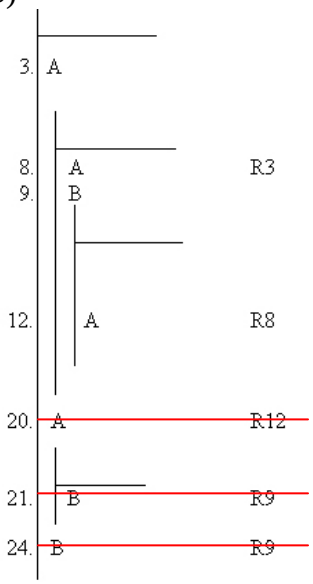


Fig. 2.

Example 4.

1.	A	Pr.
2.	C	Assp. (→Int)
3	A	R 1
4.	C → A	→Int 2-3

Ex. →Int.I.c. In all of the following proof schemata, you are asked to apply the →Int rule twice to derive a certain conditional whose consequent is also a conditional. Fill in the information missing in steps 8 and 9. (Note that the point of the exercise is not to actually construct the whole proof!)

		Pr.
3.	A	Assp. (→Int)
4.	B	Assp. (→Int)
7.	C	
8.	B → C	→ Int 4–7
9.	A → (B → C)	→ Int 3–8

		Pr.
3.	~A	Assp. (→Int)
4.	B	Assp. (→Int)
7.	~C	
8.	B → ~C	→ Int 4–7
9.	~A → (B → ~C)	→ Int 3–8

		Pr.
3.	B ∨ C	Assp. (→Int)
4.	~B	Assp. (→Int)
7.	C	
8.	~B → C	→ Int 4–7
9.	(B ∨ C) → (~B → C)	→ Int 3–8

		Pr.
3.	~B	Assp. (→Int)
4.	B ∨ C	Assp. (→Int)
7.	C	
8.	(B ∨ C) → C	→ Int 4–7
9.	~B → [(B ∨ C) → C]	→ Int 3–8

		Pr.
3.	~B	Assp. (→Int)
4.	B	Assp. (→Int)
7.	B ∨ C	
8.	B → (B ∨ C)	→ Int 4–7
9.	~B → [B → (B ∨ C)]	→ Int 3–8

		Pr.
3.	A → B	Assp. (→Int)
4.	D	Assp. (→Int)
7.	C	
8.	D → C	→ Int 4–7
9.	(A → B) → (D → C)	→ Int 3–8

Ex. →Int.I.d. In all of the following proof schemata, you are asked to apply the →Int rule twice to derive a conditional (line 9). To do that we need construct two nested subderivations, which have already been constructed. Your task is to fill in the assumptions of both subderivations (steps 3 and 4) and their respective conclusions (steps 7 and 8). (As before, the point of the exercise is not to actually construct the whole proof!)

		Pr.
3.	B	Assp. (→Int)
4.	A	Assp. (→Int)
7.	C	
8.	A → C	→ Int 4–7
9.	B → (A → C)	→ Int 3–8

		Pr.
3.	~C	Assp. (→Int)
4.	~B	Assp. (→Int)
7.	~A	
8.	~B → ~A	→ Int 4–7
9.	~C → (~B → ~A)	→ Int 3–8

		Pr.
3.	B • C	Assp. (→Int)
4.	A	Assp. (→Int)
7.	C • A	
8.	A → (C • A)	→ Int 4–7
9.	(B • C) → [A → (C • A)]	→ Int 3–8

		Pr.
3.	B	Assp. (→Int)
4.	A	Assp. (→Int)
7.	C → A	
8.	A → (C → A)	→ Int 4–7
9.	B → [A → (C → A)]	→ Int 3–8

		Pr.
3.	B	Assp. (→Int)
4.	A → C	Assp. (→Int)
7.	C	
8.	(A → C) → C	→ Int 4–7
9.	B → [(A → C) → C]	→ Int 3–8

		Pr.
3.	B → C	Assp. (→Int)
4.	A	Assp. (→Int)
7.	C	
8.	A → C	→ Int 4–7
9.	(B → C) → (A → C)	→ Int 3–8

Ex. →Int.III. Complete the following proofs. First, you have to complete the preparation of the subderivations filling in the assumptions in lines 2 and 3 and the wanted conclusion in line 5 and 6. You should then fill in the missing step in line 4 and justify all steps:

1.	$(A \bullet B) \rightarrow C$	Pr.
2.	$\quad B$	Assp. (→Int)
3.	$\quad\quad A$	Assp. (→Int)
4.	$\quad\quad A \bullet B$	•Int 2,3
5.	$\quad\quad C$	→Elim 1,4
6.	$\quad A \rightarrow C$	→Int 3–5
7.	$B \rightarrow (A \rightarrow C)$	→Int 2–6

1.	$(A \vee G) \rightarrow C$	Pr.
2.	$\quad A$	Assp. (→Int)
3.	$\quad\quad B$	Assp. (→Int)
4.	$\quad\quad A \vee G$	∨Int 2
5.	$\quad\quad C$	→Elim 1,4
6.	$\quad B \rightarrow C$	→Int 3–5
7.	$A \rightarrow (B \rightarrow C)$	→Int 2–6

1.	$(A \rightarrow C) \bullet G$	Pr.
2.	$\quad A$	Assp. (→Int)
3.	$\quad\quad B$	Assp. (→Int)
4.	$\quad\quad A \rightarrow C$	•Elim 1
5.	$\quad\quad C$	→Elim 2,4
6.	$\quad B \rightarrow C$	→Int 3–5
7.	$A \rightarrow (B \rightarrow C)$	→Int 2–6

1.	$A \rightarrow (B \rightarrow C)$	Pr.
2.	$\quad B$	Assp. (→Int)
3.	$\quad\quad A$	Assp. (→Int)
4.	$\quad\quad B \rightarrow C$	→Elim 1, 3
5.	$\quad\quad C$	→Elim 2,4
6.	$\quad A \rightarrow C$	→Int 3–5
7.	$B \rightarrow (A \rightarrow C)$	→Int 2–6

1.	$G \equiv (A \vee B)$	Pr.
2.	$\quad G$	Assp. (→Int)
3.	$\quad\quad \sim A$	Assp. (→Int)
4.	$\quad\quad A \vee B$	≡Elim 1, 2
5.	$\quad\quad B$	DS 3, 4
6.	$\quad \sim A \rightarrow B$	→Int 3–5
7.	$G \rightarrow (\sim A \rightarrow B)$	→Int 2–6

1.	$C \vee (B \vee A)$	Pr.
2.	$\quad \sim C$	Assp. (→Int)
3.	$\quad\quad \sim B$	Assp. (→Int)
4.	$\quad\quad B \vee A$	DS 1, 2
5.	$\quad\quad A$	DS 3, 4
6.	$\quad \sim B \rightarrow A$	→Int 3–5
7.	$\sim C \rightarrow (\sim B \rightarrow A)$	→Int 2–6

→Int.IV.

(a) Prove that: $B \rightarrow (A \rightarrow C)$

1.	$A \rightarrow (B \rightarrow C)$	Pr.
2.	$\quad B$	Assp (→Int)
3.	$\quad\quad A$	Assp (→Int)
4.	$\quad\quad B \rightarrow C$	→Elim 1, 3
5.	$\quad\quad C$	→Elim 2,4
6.	$\quad A \rightarrow C$	→Int 3–5
7.	$B \rightarrow (A \rightarrow C)$	→Int 2–6

(b) Prove that: $A \rightarrow (B \rightarrow C)$

1.	$B \rightarrow (A \rightarrow C)$	Pr.
2.	$\quad A$	Assp (→Int)
3.	$\quad\quad B$	Assp (→Int)
4.	$\quad\quad A \rightarrow C$	→Elim 1, 3
5.	$\quad\quad C$	→Elim 2,4
6.	$\quad B \rightarrow C$	→Int 3–5
7.	$A \rightarrow (B \rightarrow C)$	→Int 2–6

(c) Prove that: C

1.	$(A \rightarrow B) \equiv C$	Pr.
2.	$B \bullet D$	Pr.
3.	$\quad A$	Assp (→Int)
4.	$\quad B$	•Elim 2
5.	$A \rightarrow B$	→Int 3–4
6.	C	≡Elim 1, 5

(d) Prove that: $\sim A \rightarrow [\sim B \rightarrow (C \bullet D)]$

1.	$C \vee A$	Pr.
2.	$D \vee B$	Pr.
3.	$\quad \sim A$	Assp (→Int)
4.	$\quad\quad \sim B$	Assp (→Int)
5.	$\quad\quad D$	DS 2, 4
6.	$\quad\quad C$	DS 1, 3
7.	$\quad\quad C \bullet D$	•Int 6, 5
8.	$\quad \sim B \rightarrow (C \bullet D)$	→Int 4–7
9.	$\sim A \rightarrow [\sim B \rightarrow (C \bullet D)]$	→Int 3–8

(e) Prove that: C

1.	$(A \rightarrow B) \equiv C$	Pr.
2.	$(A \rightarrow A) \rightarrow B$	Pr.
3.	$\quad A$	Assp (→Int)
4.	$\quad\quad A$	Assp (→Int)
5.	$\quad\quad A$	R4
6.	$\quad A \rightarrow A$	→Int 4–5
7.	$\quad B$	→Elim 2, 6
8.	$A \rightarrow B$	→Int 3–7
9.	C	≡Elim 1, 8

(f) Prove that: $A \rightarrow (B \bullet C)$

1.	$A \rightarrow C$	Pr.
2.	$A \rightarrow B$	Pr.
3.	$\quad A$	Assp (→Int)
4.	$\quad B$	→Elim 2, 3
5.	$\quad C$	→Elim 1, 3
6.	$\quad B \bullet C$	•Int 4, 5
7.	$A \rightarrow (B \bullet C)$	→Int 3–6

(g) Prove that: $(B \rightarrow C) \rightarrow (A \rightarrow C)$

1.	$A \rightarrow B$	Pr.
2.	$\quad B \rightarrow C$	Assp ($\rightarrow$Int)
3.	$\quad\quad A$	Assp ($\rightarrow$Int)
4.	$\quad\quad B$	$\rightarrow$Elim 1, 3
5.	$\quad\quad C$	$\rightarrow$Elim 2,4
6.	$\quad A \rightarrow C$	$\rightarrow$Int 3–5
7.	$(B \rightarrow C) \rightarrow (A \rightarrow C)$	$\rightarrow$Int 2–6

(h) Prove that: $B \rightarrow (A \bullet C)$

1.	$A \bullet (B \rightarrow C)$	Pr.
2.	$\quad B$	Assp ($\rightarrow$Int)
3.	$\quad A$	•Elim 1
4.	$\quad B \rightarrow C$	•Elim 1
5.	$\quad C$	$\rightarrow$Elim 4, 2
6.	$\quad A \bullet C$	•Int 3, 5
7.	$B \rightarrow (A \bullet C)$	$\rightarrow$Int 2–6

(i) Prove that: $A \rightarrow (B \rightarrow C)$

1.	$C \bullet D$	Pr.
2.	$\quad A$	Assp ($\rightarrow$Int)
3.	$\quad\quad B$	Assp ($\rightarrow$Int)
4.	$\quad\quad C$	•Elim 1
5.	$\quad B \rightarrow C$	$\rightarrow$Int 3–4
6.	$A \rightarrow (B \rightarrow C)$	$\rightarrow$Int 2–5

(j) Prove that: $A \rightarrow (B \rightarrow (D \rightarrow C))$

1.	$C \bullet D$	Pr.
2.	$\quad A$	Assp ($\rightarrow$Int)
3.	$\quad\quad B$	Assp ($\rightarrow$Int)
4.	$\quad\quad\quad D$	Assp ($\rightarrow$Int)
5.	$\quad\quad\quad C$	•Elim 1
6.	$\quad\quad D \rightarrow C$	$\rightarrow$Int 4–5
7.	$\quad B \rightarrow (D \rightarrow C)$	$\rightarrow$Int 3–6
8.	$A \rightarrow (B \rightarrow (D \rightarrow C))$	$\rightarrow$Int 2–7

(k) Prove that: $A \rightarrow (D \rightarrow C)$

1.	$A \rightarrow (B \rightarrow C)$	Pr.
2.	$D \rightarrow B$	Pr.
3.	$\quad A$	Assp ($\rightarrow$Int)
4.	$\quad\quad D$	Assp ($\rightarrow$Int)
5.	$\quad\quad B \rightarrow C$	$\rightarrow$Elim 1, 3
6.	$\quad\quad B$	$\rightarrow$Elim 2, 4
7.	$\quad\quad C$	$\rightarrow$Elim 5, 6
8.	$\quad D \rightarrow C$	$\rightarrow$Int 4–7
9.	$A \rightarrow (D \rightarrow C)$	$\rightarrow$Int 3–8

(l) Prove that: $(A \vee B) \rightarrow C$

1.	$(A \rightarrow C) \bullet \sim B$	Pr.
2.	$(B \vee C) \vee D$	Pr.
3.	$\quad A \vee B$	Assp ($\rightarrow$Int)
4.	$\quad A \rightarrow C$	•Elim 1
5.	$\quad \sim B$	•Elim 1
6.	$\quad A$	DS 3, 5
7.	$\quad C$	$\rightarrow$Elim 4, 6
8.	$(A \vee B) \rightarrow C$	$\rightarrow$Int 3–7