Workbook Unit 11:

Natural Deduction Proofs (II)

Overview

This unit

- introduces three inference rules: ≡Elim, ∨Int, D.S.
- teaches you how to apply inference rules correctly
- teaches you how to construct relatively simple proofs using the six rules now introduced.

Prerequisites

You need to have completed Unit 10.

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1. Biconditional Elimination Rule (≡Elim)

If there is a line in the proof with a biconditional (standing on its own) and there is another line in the proof with one of its terms (standing on its own), then you are allowed to introduce another line to the proof with the other term of the biconditional (standing on its own).

i.	$p \equiv r$		*i.*	$p \equiv r$	
j.	p		*j.*	r	
➢	***r***	**≡Elim *i, j***	➢	***p***	**≡Elim *i, j***

1.1. Intuitions

The ≡Elim rule is similar to the →Elim rule but there is an important difference between them. While the ≡Elim rule has two versions, the →Elim rule has only one version – it allows you to draw the inference in only one «direction»: you need to have the antecedent of a conditional to infer the consequent of the conditional. In the case of the ≡Elim rule, you can infer both the second term of the biconditional (r) if you have the first term (*p*) and vice versa you can infer the first term (p) if you have the second term (r). This is related to the fact that we can think of the biconditional "p if and only if p" as a conjunction of two conditionals "p if r" (r → p) and "p only if r" (p → r).

The intuitiveness of the ≡Elim rule is also illustrated by a reflection on the truth-values. If we know that the biconditional p ≡ r is true and that one of its terms is true, then we also know that the other term must be true, for the biconditional is true just in case both of its terms have the same truth-value.

Complete the following arguments to be completely convinced that the ≡Elim rule is intuitive:

Ann will get a 6 on the test if and only if she gets 100% on it.
Ann got a 6 on the test.

So,

Ben will get a 6 on the test if and only if he gets 100% on it.
Ben got a 100% on the test.

So,

1.2. Applying the ≡Elim rule

Given the required two propositions, you can only apply the ≡Elim rule *in one way* (caution: the second version of the rule requires that you be given different propositions):

1.	~A ≡ ~(B • C)	Pr.
2.	~A	Pr.
3.	~(B • C)	≡Elim 1,2

1.	B ≡ (~B ∨ A)	Pr.
2.	~B ∨ A	Pr.
3.	B	≡Elim 1,2

As all inference rules, the ≡Elim rule cannot be applied to statement components:

1.	A ≡ B	Pr.
2.	A ∨ C	Pr.
~~3.~~	~~B~~	~~≡Elim 1,2~~

1.	(A ≡ B) ≡ D	Pr.
2.	B	Pr.
~~3.~~	~~A~~	~~≡Elim 1,2~~

Good advice about ≡Elim:

Do not confuse ≡Elim with →Elim

Exercise on Applying ≡Elim

Check your answers with *Solutions*. DO NOT postpone doing these exercises. Do them now!

≡Elim.I. Fill in the missing information:

1.	C ≡ D	Pr.
2.	C	Pr.
3.		≡Elim 1, 2

1.	C ≡ D	Pr.
2.	D	Pr.
3.		≡Elim 1, 2

1.	B ≡ ~D	Pr.
2.		Pr.
3.	B	≡Elim 1, 2

1.	(C ∨ A) ≡ B	Pr.
2.		Pr.
3.	B	≡Elim 1, 2

1.	A ≡ (D • B)	Pr.
2.		Pr.
3.	A	≡Elim 1, 2

1.	M ≡ ~~N	Pr.
2.		Pr.
3.	M	≡Elim 1, 2

1.	~A ≡ ~B	Pr.
2.		Pr.
3.	~B	≡Elim 1, 2

1.		Pr.
2.	~C ≡ (A • B)	Pr.
3.	~C	≡Elim 1, 2

1.	(A → B) ≡ (C ≡ D)	Pr.
2.		Pr.
3.	A → B	≡Elim 1, 2

1.	A ≡ B	Pr.
2.		Pr.
3.	B	≡Elim 1, 2

1.		Pr.
2.	(~D ≡ A) • C	Pr.
3.	~D ≡ A	Pr.
4.	~D	

1.	~A ≡ ~C	Pr.
2.	~A ≡ D	Pr.
3.		Pr.
4.	~A	≡Elim 1, 3

1.	C	Pr.
2.	A	Pr.
3.	[A ≡ (A ≡ B)] ≡ C	Pr.
4.		≡Elim 1, 3

1.	~D ≡ ~C	Pr.
2.	A ≡ C	Pr.
3.		Pr.
4.	~D	≡Elim 1, 3

1.	~(D • A)	Pr.
2.	(~D → A) ≡ C	Pr.
3.	~(D • A) ≡ ~C	Pr.
4.		≡Elim 1,3

1.	A ≡ B	Pr.
2.	B ≡ C	Pr.
3.	B	Pr.
4.		≡Elim 1, 3

1.	(A ≡ B) ≡ C	Pr.
2.	~(B ≡ C)	Pr.
3.		Pr.
4.	C	≡Elim 1, 3

1.	~A ≡ ~C	Pr.
2.	A ≡ (D → (A ≡ C))	Pr.
3.		Pr.
4.	A	≡Elim 2,3

1.3. Examples of Proofs

Try to complete each of the following proofs on your own. Then check read on.

Example 1

1.	(A • D) • B	Pr.	Prove: C
2.	C ≡ (A • D)	Pr.	

Let's think backward. Our goal is to derive statement C, which is the first term of the biconditional in line 2. We know a rule that allows us to derive the fist term of the biconditional (the ≡Elim rule), but we need to have the second term of the biconditional standing on its own line. The second term of the biconditional in line 2 is the conjunction A • D. We would usually apply the •Int rule to get a conjunction from its components. Note, however, that in this case, we are lucky. The wanted conjunction A • D is actually itself a conjunct of the conjunction in line 1. So we can apply •Elim to line 1:

3.	A • D	•Elim 1

Since we have the biconditional C ≡ (A • D) in line 2 and its second term A • D in line 3, we can derive its first term C:

4.	C	≡Elim 2, 3

This completes our proof.

Example 2.

1.	A • (B ≡ C)	Pr.	Prove: ~D
2.	A → C	Pr.	
3.	B ≡ ~D	Pr.	

A quick look at the premises is enough to see that we will have to use the information contained in premise 1, which means that we might as well apply •Elim rule to get the components of the conjunction to stand on their own lines:

4.	A	•Elim 1
5.	B ≡ C	•Elim 1

Let's think now. Our goal is to derive ~D, which occurs only in line 3 as the second term of the biconditional B ≡ ~D. We will be able to derive ~D by means of the ≡Elim rule as long as we have the other term of that binconditional (i.e. B) standing on its own line. But we do not have B.

How can we derive B? Aside from line 3, B occurs also in line 5, where it is the first term of the biconditional B ≡ C. (We do not need to think about line 1 because all the information from line 1 was transferred to lines 4 and 5.) We could derive B (by means of ≡Elim) if we had C standing on its own.

We do not have C standing on its own. Can we derive it? – Aside from line 5, C occurs also in line 2, where it is the consequent of a conditional A → C. We know a rule (→Elim) that allows us to derive the consequent of a conditional as long as we also have the antecedent of a conditional (here: A). But we are lucky to have A, so we can proceed with the proof:

6.	C	→Elim 2, 4

We now have B ≡ C (line 5) and C (line 6) standing on their own lines, so we can apply the ≡Elim rule to derive B:

7.	B	≡Elim 5, 6

Since we have B ≡ ~D (line 3) and we now also have B (line 7), both standing on their own lines, we can apply the ≡Elim rule to derive ~D:

8.	~D	≡Elim 3, 7

which is all that we needed to do.

≡Elim.II. The following proofs are missing exactly one step to prove the conclusion (on the last line). Fill in the missing step, justify it and justify the last step::

1.	(A ≡ B) • C	Pr.
2.	B	Pr.
3.		
4.	A	

1.	C ≡ B	Pr.
2.	B • ~A	Pr.
3.		
4.	C	

1.	B ≡ C	Pr.
2.	A → B	Pr.
3.	A	Pr.
4.		
5.	C	

1.	C → B	Pr.
2.	~A ≡ B	Pr.
3.	C	Pr.
4.		
5.	~A	

1.	A ≡ B	Pr.
2.	B ≡ C	Pr.
3.	A	Pr.
4.		
5.	C	

1.	A ≡ B	Pr.
2.	B ≡ C	Pr.
3.	C	Pr.
4.		
5.	A	

≡Elim.III. The following proofs are missing exactly two steps to prove the conclusion (on the last line). Fill in the missing steps, justify them and justify the last step:

1.	(A ≡ B) • C	Pr.
2.	C • A	Pr.
3.		
4.		
5.	B	

1.	(A ≡ B) • C	Pr.
2.	B • D	Pr.
3.		
4.		
5.	A	

1.	B ≡ C	Pr.
2.	C ≡ D	Pr.
3.	A • B	Pr.
4.		
5.		
6.	D	

1.	B ≡ C	Pr.
2.	A ≡ B	Pr.
3.	D • C	Pr.
4.		
5.		
6.	A	

1.	B ≡ C	Pr.
2.	(A → C) • C	Pr.
3.	A ≡ B	Pr.
4.		
5.		
6.	A	

1.	(A ≡ B) ≡ (~C • A)	Pr.
2.	~C	Pr.
3.	A	Pr.
4.		
5.		
6.	B	

≡Elim.IV. Prove that the indicated conclusion follows from the premises given:

Prove: C

1.	$A \equiv (B \equiv C)$	Pr.
2.	$A \equiv B$	Pr.
3.	A	Pr.

Prove: A

1.	$(A \equiv B) \equiv (B \equiv C)$	Pr.
2.	$B \equiv C$	Pr.
3.	C	Pr.

Prove: C

1.	$B \equiv (B \equiv C)$	Pr.
2.	$A \rightarrow (B \bullet D)$	Pr.
3.	A	Pr.

Prove: B • D

1.	$A \equiv B$	Pr.
2.	$C \equiv D$	Pr.
3.	$A \bullet C$	Pr.

Prove: A • C

1.	$A \equiv B$	Pr.
2.	$C \equiv D$	Pr.
3.	$B \bullet D$	Pr.

Prove: H

1.	$({\sim}A \bullet C) \equiv (B \vee C)$	Pr.
2.	$H \equiv (B \vee C)$	Pr.
3.	$({\sim}A \bullet D) \bullet C$	Pr.

2. The Disjunction Introduction (Addition) Rule (∨Int)

> Given any statement *p* (standing on its own) in the proof, you are allowed to introduce another line to the proof with a disjunction (standing on its own), where *p* is one of the disjuncts.
>
> | | | | | | |
> |---|---|---|---|---|---|
> | *i.* | *p* | | *i.* | *p* | |
> | ➢ | ***p*** **∨** ***r*** | **∨Int *i*** | ➢ | ***r*** **∨** ***p*** | **∨Int *i*** |

2.1. Intuitions

Consider the following true statement:

B: Boston is in the U.S.A.

Decide whether the following statements are true or false (A: Antwerp is in the U.S.A. city, C: Cincinnati is in the U.S.A.):

B ∨ ~C	❍ true	❍ false
B ∨ ~(C • B)	❍ true	❍ false
B ∨ ~~[~A ∨ ~(B → ~C)]	❍ true	❍ false
C ∨ B	❍ true	❍ false
[~(~A → B) ≡ ~~(C ∨ A)] ∨ B	❍ true	❍ false

All of these statements are true as will be any disjunction with the true statement B as a disjunct. Recall that a disjunction is true as long as at least one of its disjuncts is true. This fact about the truth-value of disjunctions constitutes the truth-functional justification for the ∨Int rule: the ∨Int rule is certainly truth-preserving – given that *p* is true, so must be any disjunction with *p* as its disjunct.

While the truth-functional justification of the rule is quite clear – there is more of a problem in finding an intuitive justification for the rule of the sort that we have been providing for all the rules thus far. Consider the following arguments:

Einstein's theory is ingenious.
____
Either Einstein's theory or Plato's theory is ingenious.

Betty Smith will come to lecture on Friday.
____
Either Betty Smith or George W. Bush will come to lecture on Friday.

You are likely not to find those arguments as intuitive as you found the arguments illustrating the other inference rules. However, you should note that the reason why we find those arguments unintuitive is *not* related to the fact that we think them *invalid*. After all, after thinking about them for a minute, we would certainly say that the conclusion cannot be false *if* the premise is true (that means that we find those arguments valid). The reason why we find those arguments unintuitive is rather that we find it hard to understand why someone who already knows that *p* is true would ever want to «dilute» this solid information into a disjunctive form "*p* or something-else." It turns out, however, that the ∨Int rule is in fact indispensable for capturing the validity of quite a number of arguments – we apply the ∨Int rule but implicitly without ever noting that we do.

2.2. Why Is the ∨Int Rule Useful?

Consider the following reasoning:

> If you get either an A or a B in logic, your parents will buy you a BMW.
> You got an A in logic.
>
> So, your parents will buy you a BMW.

Let's symbolize and set up a derivation (let A stand for "You get an A in logic", B for "You get a B in logic" and W for "Your parents will buy you a BMW"):

(A ∨ B) → W
A
―――――――
W

Let's try to prove the validity of the argument:

1.	(A ∨ B) → W	Pr.	Prove: W
2.	A	Pr.	

The wanted conclusion is in the consequent of the first premise. You could use →Elim if you had the antecedent of that conditional, i.e. if you had the disjunction A ∨ B standing on its own line. But you do not have this disjunction on its own line. The only other information you have been given is capture in line 2. You have been given A, which is one of the disjuncts of the required disjunction. Fortunately, there is a rule (the ∨Int) rule that will allow you to match the gap between the statement that you have (a) and the statement that you want (A ∨ B). For the ∨Int rule says that you are allowed to add (and create a disjunction) any statement to a statement you already have, so in particular you are allowed to add (and create a disjunction) B to A. If so, then the proof falls into place nicely:

1.	(A ∨ B) → W	Pr.	Prove: W
2.	A	Pr.	
3.	A ∨ B	∨Int 2	
4.	W	→Elim 1, 3	

Think about this derivation and about the English reasoning and you should get an idea why ∨Int is useful.

2.3. Why is ∨Int Counterintuitive?

You already know the answer to this question but it will pay to consider this point explicitly once more in the context of constructing proofs.

The ∨Int rule allows you to disjoin *any* statement with the one you already have. Consider the following two premises:

1.	$A \rightarrow C$	Pr.
2.	A	Pr.

Can you add ~A to premise 2? The answer is yes (line 3, below).
Can you add ~A to premise 1? The answer is again yes (line 4, below).
Can you add the terribly looking statement

$$[\sim(\sim A \rightarrow B) \equiv \sim\sim(C \vee A)] \bullet \sim[\sim(\sim D \equiv \sim B) \bullet \sim\sim(\sim D \vee \sim(C \vee \sim B))]$$

to line 2? The answer is again „yes" (line 5, below).
Can you add that same horrible statement to line 1? The answer is again „yes" (line 6, below).

3.	$A \vee \sim A$	∨Int 2
4.	$\sim A \vee (A \rightarrow C)$	∨Int 1
5.	$\{[\sim(\sim A \rightarrow B) \equiv \sim\sim(C \vee A)] \bullet \sim[\sim(\sim D \equiv \sim B) \bullet \sim\sim(\sim D \vee \sim(C \vee \sim B))]\} \vee A$	∨Int 2
6.	$(A \rightarrow C) \vee \{[\sim(\sim A \rightarrow B) \equiv \sim\sim(C \vee A)] \bullet \sim[\sim(\sim D \equiv \sim B) \bullet \sim\sim(\sim D \vee \sim(C \vee \sim B))]\}$	∨Int 1

The ∨Int rule will justify all the above steps as well as millions of other steps. This is one rule that you should never apply in the working-forward mode. It has be applied very carefully when you know exactly what sort of disjunction you need in the proof. Otherwise you will get lost in thousands of useless – though legal – steps.

Note that in the above case, though we were given the premises we were not given a conclusion. Suppose that we are to prove that $[(A \rightarrow C) \vee B] \bullet (D \vee A)$.

Example 3

1.	$A \rightarrow C$	Pr.	Prove: $[(A \rightarrow C) \vee B] \bullet (D \vee A)$
2.	A	Pr.	

The conclusion we need to derive is a conjunction. We will be able to derive it by means of the •Int rule as long as we have the conjuncts standing on their own lines. In other words, we need the disjunction $(A \rightarrow C) \vee B$, and another disjunction $D \vee A$. Let's consider how we can derive them in turn.

We could derive $D \vee A$ by means of the ∨Int rule if we had one of the disjuncts standing on its own line. But we do have one of the disjuncts standing on its own for we have A standing on its own in line 2. We can add any statement to A so, in particular, we can add D:

3. | $D \vee A$ ∨Int 2

It does not take much more thought to consider how to derive the other disjunction $(A \rightarrow C) \vee B$ that we want. Here too we already have one of its disjuncts standing on its own – we have $A \rightarrow C$ in line 1. We can add any statement to $A \rightarrow C$ so, in particular, we can add B:

4. | $(A \rightarrow C) \vee B$ ∨Int 1

The last step is just the application of the •Int rule to the two disjunctions:

5. | $[(A \rightarrow C) \vee B] \bullet (D \vee A)$ •Int 4, 3

Good advice about ∨Int:

Never apply ∨Int until you know exactly what you are going to do with the resulting disjunction.

2.4. Applying the ∨Int Rule

The ∨Int rule can be applied in an *unlimited* number of ways, though once we decide what statement we want to add to the one we have, it can be applied in exactly two ways. In the case below, once we decided that we want to add the statement D ≡ C to ~A, we can do this in two ways:

1.	~A	Pr.
2.	(D ≡ C) ∨ ~A	∨Int 1
3.	~A ∨ (D ≡ C)	∨Int 1

As all inference rules, the ∨Int rule cannot be applied to the statement components:

1.	~A	Pr.
~~2.~~	~~~(A ∨ (D ≡ C)~~	~~∨Int 1~~

1.	A • B	Pr.
~~2.~~	~~(A ∨ C) • B~~	~~∨Int 1~~

Exercise on Applying ∨Int

∨Int.I.a. Apply the ∨Int rule by adding statement B:

1.	A	Pr.
2.	A → C	Pr.
3.		∨Int 1
4.		∨Int 1

1.	A	Pr.
2.	A → C	Pr.
3.		∨Int 2
4.		∨Int 2

1.	~B	Pr.
2.	B → B	Pr.
3.		∨Int 2
4.		∨Int 2

1.	~B	Pr.
2.	B → B	Pr.
3.		∨Int 1
4.		∨Int 1

1.	B	Pr.
2.	A ∨ C	Pr.
3.		∨Int 1
4.		∨Int 2
5.		∨Int 2

1.	~A	Pr.
2.	A ≡ C	Pr.
3.		∨Int 1
4.		∨Int 1
5.		∨Int 2
6.		∨Int 2

$\vee$Int.I.b. Apply the $\vee$Int rule by adding statement $\sim B$:

1.	A	Pr.
2.	$A \to C$	Pr.
3.		$\vee$Int 1
4.		$\vee$Int 1

1.	A	Pr.
2.	$A \to C$	Pr.
3.		$\vee$Int 2
4.		$\vee$Int 2

1.	$\sim A$	Pr.
2.	$A \equiv C$	Pr.
3.		$\vee$Int 1
4.		$\vee$Int 1

1.	B	Pr.
2.	$A \vee C$	Pr.
3.		$\vee$Int 1
4.		$\vee$Int 1

1.	$\sim B$	Pr.
2.	$B \to B$	Pr.
3.		$\vee$Int 1

1.	$\sim B$	Pr.
2.	$B \to B$	Pr.
3.		$\vee$Int 2
4.		$\vee$Int 2

$\vee$Int.I.c. Apply the $\vee$Int rule by adding statement $\sim B \equiv A$:

1.	A	Pr.
2.	$A \to C$	Pr.
3.		$\vee$Int 1
4.		$\vee$Int 1

1.	$\sim A$	Pr.
2.	$A \equiv C$	Pr.
3.		$\vee$Int 1
4.		$\vee$Int 1

1.	A	Pr.
2.	$A \to C$	Pr.
3.		$\vee$Int 2
4.		$\vee$Int 2

1.	$\sim B$	Pr.
2.	$B \to B$	Pr.
3.		$\vee$Int 2
4.		$\vee$Int 2

2.5. Another Example of a Proof Using the ∨Int Rule

Let's do one more proof that will illustrate the use of the ∨Int rule. As always, try to do it on your before reading further.

Example 4.

1.	(C • D) ≡ (~A ∨ B)	Pr.	Prove: C
2.	B	Pr.	

Our goal is to get C standing on its own. C is a conjunct of a conjunction, which itself is the first term of the biconditional in line 1. If we somehow can derive the conjunction C • D onto its own line, we will be able to apply •Elim and get the wanted conclusion. But first we need to think about how to get C • D onto its own line.

We noticed that C • D is the first term of the biconditional. The ≡Elim rule allows us to derive the first term of a biconditional but we would need to have the second term of the biconditional (here: ~A ∨ B) on its own line, which we do not.

Though we do not have the disjunction ~A ∨ B, there is a simple way in which we can get it from B (line 2). We will simply need to apply ∨Int to B and add to it what we want. In our case, we want to add ~A to it:

3.	~A ∨ B	∨Int 2

We now have the biconditional (C • D) ≡ (~A ∨ B) in line 1 and its second term ~A ∨ B in line 3, so we can apply ≡Elim and get the first term to stand on its own:

4.	C • D	≡Elim 1, 3

All that remains is the application of the •Elim rule to get the wanted conclusion:

5.	C	•Elim 4

Proof Exercises for ∨Int

∨Int.II. The following proofs are missing exactly one step to prove the conclusion (on the last line). Fill in the missing step, justify it and justify the last step:

1.	A	Pr.
2.	(A ∨ B) → C	Pr.
3.		
4.	C	

1.	(D ∨ ~B) → A	Pr.
2.	~B	Pr.
3.		
4.	A	

1.	~B	Pr.
2.	(A ∨ ~B) → C	Pr.
3.		
4.	C	

1.	(C ∨ ~B) → (~A ∨ ~B)	Pr.
2.	C	Pr.
3.		
4.	~A ∨ ~B	

1.	A	Pr.
2.	D ≡ (A ∨ C)	Pr.
3.		
4.	D	

1.	~B	Pr.
2.	C ≡ (~A ∨ ~B)	Pr.
3.		
4.	C	

1.	(B ∨ A) ≡ (C ∨ D)	Pr.
2.	A	Pr.
3.		
4.	C ∨ D	

1.	~A	Pr.
2.	C	Pr.
3.		
4.	(C ∨ A) ∨ (C ∨ D)	

1.	~A	Pr.
2.	C	Pr.
3.		
4.	B ∨ (C ∨ D)	

1.	~A	Pr.
2.	C	Pr.
3.		
4.	(C ∨ B) ∨ D	

∨Int.III. The following proofs are missing exactly two steps to prove the conclusion (on the last line). Fill in the missing steps, justify them and justify the last step:

1.	A • B	Pr.
2.	(A ∨ C) → D	Pr.
3.		
4.		
5.	D	

1.	~D ≡ (A ∨ C)	Pr.
2.	C • B	Pr.
3.		
4.		
5.	~D	

1.	A	Pr.
2.	C	Pr.
3.		
4.		
5.	(A∨ B) • (D ∨ C)	

1.	A	Pr.
2.	A → [(A ∨ B) → D]	Pr.
3.		
4.		
5.	D	

1.	(C ∨ A) → [D ≡ (C ∨ A)]	Pr.
2.	A	Pr.
3.		
4.		
5.	D	

1.	A	Pr.
2.	(C ∨ A) → B	Pr.
3.		
4.		
5.	B ∨ C	

∨Int.IV. Prove that the indicated conclusion follows from the premises given:

Prove: ~C

1.	(A ∨ B) → D	Pr.
2.	(~E ∨ D) → ~C	Pr.
3.	A	Pr.

Prove: [(A ∨ B) ∨ C] • (D ∨ A)

1.	A	Pr.
2.	~B	Pr.

3. Disjunctive Syllogism (D.S.)

We will break with our presentation of primary inference rules of system SD and introduce one secondary, and very intuitive and useful, rule called the „Disjunctive Syllogism" (D.S.). We will later see that the rule can in fact be proven using just the primary rules in the system – this is the sense in which D.S. is a secondary rule.

> If there is a line in the proof with a disjunction (standing on its own) and there is another line in the proof with the negation of one of its disjuncts (standing on its own), then you are allowed to introduce another line to the proof with the other disjunct (standing on its own).
>
> | | | | | | | |
> |---|---|---|---|---|---|---|
> | *i.* | $p \vee r$ | | | *i.* | $p \vee r$ | |
> | *j.* | $\sim p$ | | | *j.* | $\sim r$ | |
> | ➤ | r | **D.S. *i, j*** | | ➤ | p | **D.S. *i, j*** |

3.1. Intuitions

The D.S. rule is extremely intuitive. Fill in the conclusions of the above arguments to convince yourself that it is so:

There is either ice-cream or pudding for desert in the dining hall.
Unfortunately, there is no longer any ice-cream left.

Alice has either a dog or a cat.
Alice does not have a cat.

3.2. Applying the D.S. rule

Each version of the two versions of the D.S. rule can be applied only in one way:

1.	A ∨ B	Pr.		1.	A ∨ B	Pr.
2.	~A	Pr.		2.	~B	Pr.
3.	B	D.S. 1, 2		3.	A	D.S. 1, 2

The more tricky applications of D.S. will be those where at least one of the terms of the disjunction is a negation. Think for a moment about the following cases and fill in the missing information:

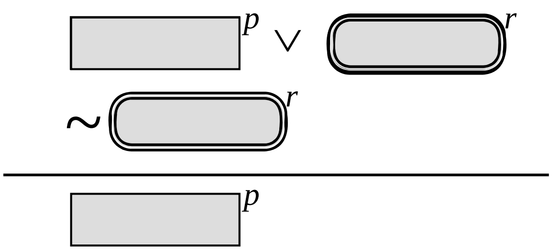
1.	D ∨ ~C	Pr.		1.	D ∨ ~C	Pr.
2.	~D	Pr.		2.		Pr.
3.		D.S. 1, 2		3.	D	D.S. 1, 2

In the first case we are given a disjunction D ∨ ~C and a negation ~D. In this case, D.S. allows to infer the second disjunct (~D is a negation of the first disjunct D), i.e. it allows us to infer ~C.

In the second case, we are given the same disjunction D ∨ ~C but this time we are asked what else we would need to have to derive the first disjunct D. It is here that most students make a mistake in applying the D.S. and think that what we would need to have is C.

1.	D ∨ ~C	Pr.
~~2.~~	~~C~~	~~Pr.~~
3.	D	D.S. 1, 2

This is a mistake in applying the inference rule. For the inference rule tells us that we given a disjunction *p* ∨ *r*, we can derive the first disjunct *p* as long as we have the negation of the second disjunct, i.e. as long as we have ~*r*. Fill in the variables in the following schema (D needs to jump into the *p*-box while ~C needs to jump into the *r*-box:



You can now see why the rule can only be applied in the following way:

1.	D ∨ ~C	Pr.
2.	~~C	Pr.
3.	D	D.S. 1, 2

Good advice about D.S.:

Watch out for negations!

As all inference rules, the D.S. rule cannot be applied to statement components:

1.	A ∨ (B ∨ C)	Pr.
2.	~B	Pr.
~~3.~~	~~C~~	~~D.S. 1, 2~~

1.	A ∨ B	Pr.
2.	~A ≡ C	Pr.
~~3.~~	~~B~~	~~D.S. 1,2~~

Exercises on Applying D.S.

D.S.I. Fill in the missing information:

1.	A ∨ B	Pr.
2.	~B	Pr.
3.		D.S. 1,2

1.	A ∨ B	Pr.
2.		Pr.
3.	B	D.S. 1,2

1.		Pr.
2.	~B	Pr.
3.	C	D.S. 1,2

1.	~A ∨ ~B	Pr.
2.		Pr.
3.	~A	D.S. 1,2

1.		Pr.
2.	~(B • C)	Pr.
3.	~A	D.S. 1,2

1.	(A → C) ∨ B	Pr.
2.	~B	Pr.
3.		D.S. 1,2

1.	A ∨ ~B	Pr.
2.		Pr.
3.	A	D.S. 1,2

1.	~A	Pr.
2.		Pr.
3.	~~B	D.S. 1,2

1.	~C ∨ B	Pr.
2.	~B	Pr.
3.		D.S. 1,2

1.	~C ∨ B	Pr.
2.	~~C	Pr.
3.		D.S. 1,2

1.	~D ∨ ~A	Pr.
2.	~~A	Pr.
3.		D.S. 1,2

1.	~C ∨ ~D	Pr.
2.		Pr.
3.	~C	D.S. 1,2

1.		Pr.
2.	~(B • C)	Pr.
3.	~A	D.S. 1,2

1.	~A ∨ (B • C)	Pr.
2.		Pr.
3.	B • C	D.S. 1,2

1.	~A	Pr.
2.	~~B ∨ A	Pr.
3.		D.S. 1,2

1.	~A	Pr.
2.	~~B	Pr.
3.	A ∨ ~B	Pr.
4.		D.S. 1,3
5.		D.S. 2,3

3.3. Examples of Proofs Using D.S.

Example 5.

Prove that C from the following premises:

1.	~A	Pr.
2.	A ∨ (B • C)	Pr.

Think about strategy. You are to obtain C. C is only present in premise 2 but it is hidden as a component of a component of premise 2. It is the second conjunct of a conjunction, which itself is the second disjunct. If you had B • C on its own, you could apply •Elim and get C out, but you don't have B • C on its own. Can you get it?

Well, the conjunction B • C is the second disjunct of the disjunction in line 2. We now know a rule (D.S.) that allows us to infer the second disjunct *if* we have the negation of the first disjunct. Our first disjunct is A, so we must have ~A to apply D.S. Luckily, we do have ~A on line 1.

So, since we have the disjunction A ∨ (B • C) in line 2 and ~A, which is the negation of its first disjunct, in line 1, we can apply D.S. to get the second disjunct:

3.	B • C	D.S. 1,2

All that remains is to apply •Elim to get C:

4.	C	•Elim 3

Example 6:

Prove that A from the following premises:

1.	~B • ~C	Pr.
2.	B ∨ (A ∨ C)	Pr.

Think about strategy. A is hidden in premise 2. In order to do anything with it, we need to get the disjunct A ∨ C on its own line. D.S. is exactly what we need, because it is a rule that allows us to derive a disjunct provided that we have the negation of the other disjunct, here we would need ~B. We don't have ~B on its own, but we can derive it from line 1 using •Elim. Let's do that:

3.	~B	•Elim 1
4.	A ∨ C	D.S. 2,3

Ultimately, we want to derive A, which is now the first disjunct of the disjunction A ∨ C in line 4. We could get A by means of D.S., if we had the negation of the second disjunct, i.e. if we had ~C. We don't have ~C but we can get it by applying •Elim to line 1:

5.	~C	•Elim 1
6.	A	D.S. 4,5

Example 7

Prove that D from the following premises:

1.	~A	Pr.
2.	~B ∨ A	Pr.
3.	B ∨ (A ∨ D)	Pr.

All you have here are disjunctions (some hidden within disjuncts) and a negation in line 1. The conclusion D is hidden in a disjunction within a disjunction (premise 3). If we were able to get the hidden disjunction (A ∨ D) to stand on its own line, we could use premise 1 (~A) to get the wanted conclusion (D) by D.S. So the main subtask is to get (A ∨ D) to stand on its own line. We could derive it by D.S. if we had ~B. We don't. Could we get ~B? Yes, we could get it by D.S. if we had… ~A… But we do have ~A in line 1. So, ready, steady and go:

We have the disjunction ~B ∨ A in line 2 and the negation of its second disjunct in line 1 (~A), so we can derive the first disjunct:

4.	~B	D.S. 1,2

We now have the disjunction B ∨ (A ∨ D) in line 3 and the negation of its first disjunct in line 4 (~B), so we can derive the second disjunct:

5.	A ∨ D	D.S. 3,4

We now have the disjunction A ∨ D in line 5 and the negation of its first disjunct in line 1 (~A), so we can derive the second disjunct:

6.	D	D.S. 1,5

Exercises on Proofs with D.S.

D.S.II. The following proofs are missing exactly one step to prove the conclusion (on the last line). Fill in the missing step, justify it and justify the last step:

1.	~D	Pr.
2.	(C ∨ D) ∨ D	Pr.
3.		
4.	C	

1.	~A • ~B	Pr.
2.	B ∨ D	Pr.
3.		
4.	D	

1.	~B	Pr.
2.	~B → (A ∨ B)	Pr.
3.		
4.	A	

1.	(~B ∨ ~A) ≡ ~~A	Pr.
2.	~~A	Pr.
3.		
4.	~B	

D.S.III. The following proofs are missing exactly two steps to prove the conclusion (on the last line). Fill in the missing steps, justify them and justify the last step:

1.	~D ≡ (A ∨ D)	Pr.
2.	~D • B	Pr.
3.		
4.		
5.	A	

1.	[~A → (D ∨ A)] ∨ A	Pr.
2.	~A	Pr.
3.		
4.		
5.	D	

1.	[(A ∨ B) ∨ C] ∨ C	Pr.
2.	~C • ~B	Pr.
3.		
4.		
5.	A ∨ B	

1.	B ∨ (A ∨ B)	Pr.
2.	~B	Pr.
3.		
4.		
5.	D ∨ A	

D.S.IV. Prove that the indicated conclusion follows from the premises given:

(a) Prove that C

1.	$D \to (A \vee C)$	Pr.
2.	$D \bullet \sim A$	Pr.

(b) Prove that ~Z

1.	$F \to (G \bullet \sim H)$	Pr.
2.	$\sim Z \vee H$	Pr.
3.	F	Pr.

(c) Prove: B

1.	C	Pr.
2.	$\sim A \equiv (\sim D \vee C)$	Pr.
3.	$(C \vee D) \to (A \vee B)$	Pr.

(d) Prove: ~B

1.	$(D \vee A) \vee [A \vee (\sim B \vee A)]$	Pr.
2.	$\sim A \to \sim(D \vee A)$	Pr.
3.	$\sim A \bullet C$	Pr.

(e) Prove that C

1.	$(\sim A \bullet \sim B) \to C$	Pr.
2.	$(\sim A \vee D) \bullet (\sim B \vee D)$	Pr.
3.	$\sim D$	Pr.

(f) Prove that Z

1.	$(A \vee B) \vee (\sim T \vee W)$	Pr.
2.	$(T \vee Z) \vee (A \vee B)$	Pr.
3.	$\sim W \bullet \sim(A \vee B)$	Pr.

(g) Prove that ~D

1.	(A • B) → ~C	Pr.
2.	C ∨ ~D	Pr.
3.	(A ∨ G) → B	Pr.
4.	E • A	Pr.

(h) Prove that ~H

1.	F → (G → ~H)	Pr.
2.	(F • ~W) → (G ∨ T)	Pr.
3.	F • ~T	Pr.
4.	~W ∨ T	Pr.

(i) Prove that R

1.	P ≡ (Q → (R ∨ S))	Pr.
2.	P • Q	Pr.
3.	~S ∨ T	Pr.
4.	~T ∨ ~W	Pr.
5.	~~W	Pr.

(j) Prove that ~D

1.	A → (~B ∨ C)	Pr.
2.	~B → (F ∨ G)	Pr.
3.	(G • ~H) → (~D ∨ B)	Pr.
4.	(A • ~C) ∨ H	Pr.
5.	~H • ~F	Pr.

What You Need to Know and Do

- You need to know the inference rules and be able to apply them
- You need to be able to construct proofs using the inference rules