

Solutions to Workbook Exercises

Unit 10:

Natural Deduction Proofs (A)

Ex. •Int-I. Please, fill in the missing information:

1.	C	Pr.
2.	D	Pr.
3.	C • D	•Int 1, 2
4.	D • C	•Int 2, 1

1.	A	Pr.
2.	B	Pr.
3.	A • B	•Int 1, 2
4.	B • A	•Int 2, 1

1.	B	Pr.
2.	C	Pr.
3.	B • C	•Int 1, 2
4.	C • B	•Int 2, 1

1.	~C	Pr.
2.	B	Pr.
3.	~C • B	•Int 1, 2
4.	B • ~C	•Int 2, 1

1.	C	Pr.
2.	A ≡ B	Pr.
3.	C • (A ≡ B)	•Int 1, 2
4.	(A ≡ B) • C	•Int 2, 1

1.	C • A	Pr.
2.	B	Pr.
3.	(C • A) • B	•Int 1, 2
4.	B • (C • A)	•Int 2, 1

1.	~A	Pr.
2.	~~B	Pr.
3.	~A • ~~B	•Int 1, 2
4.	~~B • ~A	•Int 2, 1

1.	C ∨ A	Pr.
2.	D	Pr.
3.	(C ∨ A) • D	•Int 1, 2
4.	D • (C ∨ A)	•Int 2, 1

1.	~B	Pr.
2.	C → D	Pr.
3.	~B • (C → D)	•Int 1, 2
4.	(C → D) • ~B	•Int 2, 1

Ex. •Elim-I. Please, fill in the missing information:

1.	A • B	Pr.
2.	A	•Elim 1
3.	B	•Elim 1

1.	C • D	Pr.
2.	C	•Elim 1
3.	D	•Elim 1

1.	A • C	Pr.
2.	A	•Elim 1
3.	C	•Elim 1

1.	C • B	Pr.
2.	B ≡ D	Pr.
3.	B	•Elim 1

1.	C ∨ A	Pr.
2.	A • B	Pr.
3.	A	•Elim 2

1.	(C ∨ A) • D	Pr.
2.	D → B	Pr.
3.	D	•Elim 1

1.	~A • C	Pr.
2.	B • ~D	Pr.
3.	~A	•Elim 1
4.	C	•Elim 1
5.	B	•Elim 2
6.	~D	•Elim 2

1.	A • (C ∨ B)	Pr.
2.	(B ≡ D) • D	Pr.
3.	B ≡ D	•Elim 2
4.	D	•Elim 2
5.	A	•Elim 1
6.	C ∨ B	•Elim 1

1.	(A → C) • ~D	Pr.
2.	(B • A) • C	Pr.
3.	A → C	•Elim 1
4.	~D	•Elim 1
5.	B • A	•Elim 2
6.	C	•Elim 2

Ex. •Int•Elim-II. The following proofs are missing exactly one step to prove the conclusion (on the last line). Please, fill in the missing step, justify it and justify the last step:

1.	A	Pr.
2.	B	Pr.
3.	$A \bullet B$	•Int 1, 2
4.	$(A \bullet B) \bullet B$	•Int 3, 2

1.	A	Pr.
2.	B	Pr.
3.	$B \bullet A$	•Int 2, 1
4.	$A \bullet (B \bullet A)$	•Int 1, 3

1.	A	Pr.
2.	B	Pr.
3.	$A \bullet B$	•Int 1,2
4.	$B \bullet (A \bullet B)$	•Int 2,3

1.	$C \bullet A$	Pr.
2.	B	Pr.
3.	A	•Elim 1
4.	$A \bullet B$	•Int 2, 3

1.	C	Pr.
2.	B	Pr.
3.	$B \bullet C$	•Int 1, 2
4.	$(B \bullet C) \bullet B$	•Int 3, 2

1.	C	Pr.
2.	$A \bullet B$	Pr.
3.	A	•Elim 2
4.	$A \bullet C$	•Int 1,3

1.	$\sim C$	Pr.
2.	$\sim D$	Pr.
3.	$\sim C \bullet \sim D$	•Int 1, 2
4.	$(\sim C \bullet \sim D) \bullet \sim C$	•Int 3, 1

1.	$C \bullet \sim D$	Pr.
2.	$A \rightarrow B$	Pr.
3.	$\sim D$	•Elim 1
4.	$\sim D \bullet (A \rightarrow B)$	•Int 2, 3

1.	$(A \bullet C) \bullet D$	Pr.
2.	$A \bullet C$	•Elim 1
3.	A	•Elim 2

•Int•Elim-III. The following proofs are missing exactly two steps to prove the conclusion (on the last line). Please, fill in the missing steps, justify them and justify the last step.

1.	$A \bullet B$	Pr.
2.	$C \bullet D$	Pr.
3.	A	•Elim 1
4.	D	•Elim 2
5.	$A \bullet D$	•Int 3,4

1.	$(A \bullet B) \bullet C$	Pr.
2.	D	Pr.
3.	$A \bullet B$	•Elim 1
4.	A	•Elim 3
5.	$A \bullet D$	•Int 4,2

1.	$C \bullet [A \bullet (B \bullet D)]$	Pr.
2.	$A \bullet (B \bullet D)$	•Elim 1
3.	$B \bullet D$	•Elim 2
4.	B	•Elim 3

1.	$C \bullet A$	Pr.
2.	A	•Elim 1
3.	C	•Elim 1
4.	$A \bullet C$	•Int 2,3

1.	$(A \bullet B) \bullet (C \vee D)$	Pr.
2.	$\sim G$	Pr.
3.	$A \bullet B$	•Elim 1
4.	B	•Elim 3
5.	$\sim G \bullet B$	•Int 2,4

1.	A	Pr.
2.	B	Pr.
3.	$A \bullet B$	•Int 1,2
4.	$B \bullet A$	•Int 2,1
5.	$(A \bullet B) \bullet (B \bullet A)$	•Int 3,4

•**Int•Elim-IV.** Prove that the indicated conclusion follows from the premises given (you need to determine how many steps are necessary to prove the conclusion):

a. Prove that $(A \bullet C) \bullet (C \bullet A)$

1.	A	Pr.
2.	C	Pr.
3.	$C \bullet A$	•Int 2, 1
4.	$A \bullet C$	•Int 1, 2
5.	$(A \bullet C) \bullet (C \bullet A)$	•Int 4, 3

b. Prove that $(A \bullet C) \bullet (C \bullet A)$

1.	$A \bullet B$	Pr.
2.	$B \bullet C$	Pr.
3.	A	•Elim 1
4.	C	•Elim 2
5.	$A \bullet C$	•Int 3,4
6.	$C \bullet A$	•Int 4,3
7.	$(A \bullet C) \bullet (C \bullet A)$	•Int 5,6

c. Prove that $(B \bullet C) \bullet (A \equiv D)$

1.	$(A \bullet B) \bullet \sim C$	Pr.
2.	$(C \bullet D) \bullet (A \equiv D)$	Pr.
3.	$A \equiv D$	•Elim 2
4.	$A \bullet B$	•Elim 1
5.	B	•Elim 4
6.	$C \bullet D$	•Elim 2
7.	C	•Elim 6
8.	$B \bullet C$	•Int 5,7
9.	$(B \bullet C) \bullet (A \equiv D)$	•Int 8,3

d. Prove that $A \bullet [B \bullet (A \bullet C)]$

1.	$(A \bullet B) \bullet C$	Pr.
2.	$A \bullet B$	•Elim 1
3.	C	•Elim 1
4.	A	•Elim 2
5.	B	•Elim 2
6.	$A \bullet C$	•Int 4,3
7.	$B \bullet (A \bullet C)$	•Int 5,6
8.	$A \bullet [B \bullet (A \bullet C)]$	•Int 4,7

→**Elim-I.** Please, fill in the missing information:

1.	$C \rightarrow D$	Pr.
2.	C	Pr.
3.	D	→Elim 1, 2

1.	$A \rightarrow D$	Pr.
2.	A	Pr.
3.	D	→Elim 1, 2

1.	$B \rightarrow D$	Pr.
2.	B	Pr.
3.	D	→Elim 1, 2

1.	$D \rightarrow A$	Pr.
2.	D	Pr.
3.	A	→Elim 1, 2

1.	$C \rightarrow D$	Pr.
2.	C	Pr.
3.	D	→Elim 1, 2

1.	$D \rightarrow C$	Pr.
2.	D	Pr.
3.	C	→Elim 1, 2

1.	$\sim B \rightarrow A$	Pr.
2.	$\sim B$	Pr.
3.	A	→Elim 1, 2

1.	$\sim A \rightarrow B$	Pr.
2.	$\sim A$	Pr.
3.	B	→Elim 1, 2

1.	$A \rightarrow \sim B$	Pr.
2.	A	Pr.
3.	$\sim B$	→Elim 1, 2

1.	$\sim C \rightarrow D$	Pr.
2.	$\sim C$	Pr.
3.	D	→Elim 1, 2

1.	$\sim A \rightarrow \sim D$	Pr.
2.	$\sim A$	Pr.
3.	$\sim D$	→Elim 1, 2

1.	$B \rightarrow \sim B$	Pr.
2.	B	Pr.
3.	$\sim B$	→Elim 1, 2

1.	B	Pr.
2.	$B \rightarrow \sim D$	Pr.
3.	$\sim D$	→Elim 1, 2

1.	D	Pr.
2.	$D \rightarrow B$	Pr.
3.	B	→Elim 1, 2

1.	$\sim\sim A$	Pr.
2.	$\sim\sim A \rightarrow C$	Pr.
3.	C	→Elim 1, 2

→**Elim-II**. The following proofs are missing exactly one step to prove the conclusion (on the last line). Please, fill in the missing step, justify it and justify the last step:

1.	$B \bullet C$	Pr.
2.	$B \rightarrow A$	Pr.
3.	B	•Elim 1
4.	A	→Elim 2,3

1.	C	Pr.
2.	$(C \rightarrow B) \bullet A$	Pr.
3.	$C \rightarrow B$	•Elim 2
4.	B	→Elim 1,3

1.	$(B \rightarrow A) \bullet C$	Pr.
2.	B	Pr.
3.	$B \rightarrow A$	•Elim 1
4.	A	→Elim 2,3

1.	$A \rightarrow D$	Pr.
2.	$D \rightarrow C$	Pr.
3.	A	Pr.
4.	D	→Elim 1,3
5.	C	→Elim 2,4

1.	$B \rightarrow C$	Pr.
2.	$A \rightarrow B$	Pr.
3.	A	Pr.
4.	B	→Elim 2,3
5.	C	→Elim 1,4

1.	$B \rightarrow \sim C$	Pr.
2.	$\sim A \rightarrow B$	Pr.
3.	$\sim A$	Pr.
4.	B	→Elim 2,3
5.	$\sim C$	→Elim 1,4

1.	A	Pr.
2.	B	Pr.
3.	$(A \bullet B) \rightarrow C$	Pr.
4.	$A \bullet B$	•Int 1,2
5.	C	→Elim 3,4

1.	$B \rightarrow (C \bullet A)$	Pr.
2.	$B \bullet D$	Pr.
3.	B	•Elim 2
4.	$C \bullet A$	→Elim 1,3

1.	$B \rightarrow (B \rightarrow A)$	Pr.
2.	B	Pr.
3.	$B \rightarrow A$	→Elim 1,2
4.	A	→Elim 2,3

→**Elim-III**. The following proofs are missing exactly two steps to prove the conclusion (on the last line). Please, fill in the missing steps, justify them and justify the last step:

1.	$(A \rightarrow B) \bullet C$	Pr.
2.	$C \bullet A$	Pr.
3.	$A \rightarrow B$	•Elim 1
4.	A	•Elim 2
5.	B	→Elim 3,4

1.	$A \rightarrow B$	Pr.
2.	$C \bullet (A \bullet D)$	Pr.
3.	$A \bullet D$	•Elim 2
4.	A	•Elim 3
5.	B	→Elim 1,4

1.	$B \rightarrow C$	Pr.
2.	$C \rightarrow D$	Pr.
3.	$A \bullet B$	Pr.
4.	B	•Elim 3
5.	C	→Elim 1,4
6.	D	→Elim 2,5

1.	$B \rightarrow C$	Pr.
2.	$A \rightarrow B$	Pr.
3.	A	Pr.
4.	B	→Elim 2,3
5.	C	→Elim 1,4
6.	$B \bullet C$	•Int 4,5

1.	$C \rightarrow A$	Pr.
2.	$(A \rightarrow C) \bullet C$	Pr.
3.	$A \rightarrow B$	Pr.
4.	C	•Elim 2
5.	A	→Elim 1,4
6.	B	→Elim 3,5

1.	$\sim D \rightarrow B$	Pr.
2.	$\sim A \rightarrow \sim D$	Pr.
3.	$\sim A$	Pr.
4.	$\sim D$	→Elim 2,3
5.	B	→Elim 1,4
6.	$\sim D \bullet B$	•Int 4,5

1.	C	Pr.
2.	$C \rightarrow B$	Pr.
3.	$C \rightarrow D$	Pr.
4.	B	→Elim 1,2
5.	D	→Elim 1,3
6.	$B \bullet D$	•Int 4,5

1.	$C \bullet A$	Pr.
2.	$A \rightarrow (D \bullet C)$	Pr.
3.	A	•Elim 1
4.	$D \bullet C$	→Elim 2,3
5.	D	•Elim 4

1.	$\sim D \bullet A$	Pr.
2.	$(\sim D \bullet B) \rightarrow C$	Pr.
3.	B	Pr.
4.	$\sim D$	•Elim 1
5.	$\sim D \bullet B$	•Int 4, 3
6.	C	→Elim 2, 5

→**Elim-IV**. Prove that the indicated conclusion follows from the premises given:

Prove that $A \bullet C$:

1.	$(A \rightarrow B) \bullet (B \rightarrow C)$	Pr.
2.	A	Pr.
3.	$A \rightarrow B$	•Elim 1
4.	$B \rightarrow C$	•Elim 1
5.	B	→Elim 2, 3
6.	C	→Elim 4, 5
7.	$A \bullet C$	•Int 2, 6

Prove that $B \bullet D$:

1.	$A \rightarrow B$	Pr.
2.	$C \rightarrow D$	Pr.
3.	$A \bullet C$	Pr.
4.	A	•Elim 3
5.	C	•Elim 3
6.	B	→Elim 1, 4
7.	D	→Elim 2, 5
8.	$B \bullet D$	•Int 6, 7

Ex. Components. Please, carefully inspect the following derivations and mark those that are not permitted, justify those steps that are permitted ('XXXX' means that the step is not permitted):

1.	$C \rightarrow D$	Pr.
2.	$C \bullet A$	Pr.
3.	D	XXXX

1.	$A \rightarrow D$	Pr.
2.	$A \rightarrow B$	Pr.
3.	B	XXXX

1.	$\sim C$	Pr.
2.	$\sim C \rightarrow D$	Pr.
3.	D	→Elim 1, 2

1.	$A \rightarrow B$	Pr.
2.	$C \bullet A$	Pr.
3.	A	•Elim 2
4.	B	→Elim 1, 3

1.	$B \rightarrow C$	Pr.
2.	$A \bullet (B \bullet C)$	Pr.
3.	A	•Elim 2
4.	B	XXXX

1.	$A \rightarrow D$	Pr.
2.	$A \rightarrow B$	Pr.
3.	B	XXXX
4.	D	XXXX

1.	$(A \bullet B) \bullet C$	Pr.
2.	$C \rightarrow D$	Pr.
3.	A	XXXX
4.	B	XXXX

1.	$(B \rightarrow C) \bullet D$	Pr.
2.	B	Pr.
3.	C	XXXX

1.	$A \rightarrow D$	Pr.
2.	A	XXXX

1.	$(A \bullet B) \vee C$	Pr.
2.	$C \rightarrow D$	Pr.
3.	A	XXXX
4.	B	XXXX

1.	$(A \bullet C) \bullet C$	Pr.
2.	C	•Elim 1

1.	$\sim C \bullet B$	Pr.
2.	$\sim C \rightarrow D$	Pr.
3.	$\sim C$	•Elim 1
4.	B	•Elim 1

Ex. \rightarrow Elim-2.I.

1. $(C \vee A) \rightarrow B$ Pr.
 2. $C \vee A$ Pr.
 3. B \rightarrow Elim 1, 2

1. $A \rightarrow (D \cdot B)$ Pr.
 2. A Pr.
 3. $D \cdot B$ \rightarrow Elim 1, 2

1. $M \rightarrow \sim\sim N$ Pr.
 2. M Pr.
 3. $\sim\sim N$ \rightarrow Elim 1, 2

1. $\sim\sim C \rightarrow D$ Pr.
 2. $\sim\sim C$ Pr.
 3. D \rightarrow Elim 1, 2

1. $\sim C$ Pr.
 2. $\sim C \rightarrow (A \cdot B)$ Pr.
 3. $A \cdot B$ \rightarrow Elim 1, 2

1. $(A \rightarrow B) \rightarrow (C \rightarrow D)$ Pr.
 2. $A \rightarrow B$ Pr.
 3. $C \rightarrow D$ \rightarrow Elim 1, 2

1. $\sim D$ Pr.
 2. $(\sim D \rightarrow A) \cdot C$ Pr.
 3. $\sim D \rightarrow A$ Pr.
 4. A \rightarrow Elim 1, 3

1. $\sim A \rightarrow \sim C$ Pr.
 2. $\sim A \rightarrow D$ Pr.
 3. $\sim A$ Pr.
 4. $\sim C$ \rightarrow Elim 1, 3

1. $A \rightarrow (A \rightarrow B)$ Pr.
 2. A Pr.
 3. $[A \rightarrow (A \rightarrow B)] \rightarrow C$ Pr.
 4. C \rightarrow Elim 1, 3

1. $A \rightarrow B$ Pr.
 2. $A \vee C$ Pr.
 3. A Pr.
 4. B \rightarrow Elim 1, 3

1. $\sim D \rightarrow \sim C$ Pr.
 2. $A \equiv C$ Pr.
 3. $\sim D$ Pr.
 4. $\sim C$ \rightarrow Elim 1, 3

1. $\sim(D \cdot A)$ Pr.
 2. $(\sim D \rightarrow A) \rightarrow C$ Pr.
 3. $\sim(D \cdot A) \rightarrow \sim C$ Pr.
 4. $\sim C$ \rightarrow Elim 1, 3

1. $A \rightarrow B$ Pr.
 2. $B \rightarrow C$ Pr.
 3. A Pr.
 4. B \rightarrow Elim 1, 3

1. $(A \rightarrow B) \rightarrow C$ Pr.
 2. $\sim(B \vee C)$ Pr.
 3. $A \rightarrow B$ Pr.
 4. C \rightarrow Elim 1, 3

1. $\sim A \rightarrow \sim C$ Pr.
 2. $A \rightarrow (D \rightarrow (A \rightarrow C))$ Pr.
 3. A Pr.
 4. $D \rightarrow (A \rightarrow C)$ \rightarrow Elim 2, 3

Ex. \rightarrow Elim. Int. \bullet Elim-2.II. The following proofs are missing exactly one step to prove the conclusion (on the last line). Please, fill in the missing step, justify it and justify the last step:

1. $(C \cdot A) \rightarrow D$ Pr.
 2. $D \cdot (C \cdot A)$ Pr.
 3. $C \cdot A$ \bullet Elim 2
 4. D \rightarrow Elim 1, 3

1. $(A \cdot B) \rightarrow C$ Pr.
 2. A Pr.
 3. B Pr.
 4. $A \cdot B$ \bullet Int 2, 3
 5. C \rightarrow Elim 1, 4

1. $(A \rightarrow B) \rightarrow C$ Pr.
 2. $(A \rightarrow B) \cdot D$ Pr.
 3. $A \rightarrow B$ \bullet Elim 2
 4. C \rightarrow Elim 1, 3

1. $A \rightarrow [(B \rightarrow C) \rightarrow D]$ Pr.
 2. A Pr.
 3. $B \rightarrow C$ Pr.
 4. $(B \rightarrow C) \rightarrow D$ \rightarrow Elim 1, 2
 5. D \rightarrow Elim 3, 4

1. $(A \rightarrow B) \rightarrow (A \rightarrow C)$ Pr.
 2. $A \rightarrow B$ Pr.
 3. A Pr.
 4. $A \rightarrow C$ \rightarrow Elim 1, 2
 5. C \rightarrow Elim 3, 4

Ex. \rightarrow Elim. \bullet Int. \bullet Elim-2.III. The following proofs are missing exactly two steps to prove the conclusion (on the last line). Please, fill in the missing steps, justify them and justify the last step:

1.	$(B \bullet D) \rightarrow (D \rightarrow C)$	Pr.
2.	$A \bullet (B \bullet D)$	Pr.
3.	D	Pr.
4.	$B \bullet D$	\bullet Elim 2
5.	$D \rightarrow C$	\rightarrow Elim 1,4
6.	C	\rightarrow Elim 3,5

1.	$(\sim A \bullet \sim B) \rightarrow (\sim B \rightarrow C)$	Pr.
2.	$\sim B$	Pr.
3.	$\sim A$	Pr.
4.	$\sim A \bullet \sim B$	\bullet Int 3,2
5.	$\sim B \rightarrow C$	\rightarrow Elim 1,4
6.	C	\rightarrow Elim 2,5

1.	$A \rightarrow [A \rightarrow (A \rightarrow C)]$	Pr.
2.	A	Pr.
3.	$A \rightarrow (A \rightarrow C)$	\rightarrow Elim 1, 2
4.	$A \rightarrow C$	\rightarrow Elim 2,3
5.	C	\rightarrow Elim 2,4

1.	$(A \bullet B) \rightarrow (C \bullet D)$	Pr.
2.	$(A \bullet B) \bullet G$	Pr.
3.	$A \bullet B$	\bullet Elim 2
4.	$C \bullet D$	\rightarrow Elim 1, 3
5.	D	\bullet Elim 4

1.	$(K \rightarrow L) \rightarrow (K \rightarrow M)$	Pr.
2.	$K \rightarrow L$	Pr.
3.	K	Pr.
4.	$K \rightarrow M$	\rightarrow Elim 1,2
5.	M	\rightarrow Elim 3,4
6.	$K \bullet M$	\bullet Int 3, 5

1.	$K \rightarrow [(L \rightarrow M) \rightarrow N]$	Pr.
2.	K	Pr.
3.	$L \rightarrow M$	Pr.
4.	$(L \rightarrow M) \rightarrow N$	\rightarrow Elim 1, 2
5.	N	\rightarrow Elim 3, 4
6.	$(L \rightarrow M) \bullet N$	\bullet Int 3, 5

Ex. \rightarrow Elim. \bullet Int. \bullet Elim-2.IV. Prove that the indicated conclusion follows from the premises given:

Prove that C from:

1.	$A \rightarrow (B \rightarrow C)$	Pr.
2.	$A \rightarrow B$	Pr.
3.	A	Pr.
4.	B	\rightarrow Elim 2,3
5.	$B \rightarrow C$	\rightarrow Elim 1,3
6.	C	\rightarrow Elim 4,5

Prove that A from:

1.	$(\sim C \rightarrow B) \rightarrow (B \rightarrow A)$	Pr.
2.	$\sim C$	Pr.
3.	$\sim C \rightarrow B$	Pr.
4.	B	\rightarrow Elim 2,3
5.	$B \rightarrow A$	\rightarrow Elim 1,3
6.	A	\rightarrow Elim 4,5

Prove that C from:

1.	$B \rightarrow (B \rightarrow C)$	Pr.
2.	$A \rightarrow (B \bullet D)$	Pr.
3.	A	Pr.
4.	$B \bullet D$	\rightarrow Elim 2,3
5.	B	\bullet Elim 4
6.	$B \rightarrow C$	\rightarrow Elim 1,5
7.	C	\rightarrow Elim 5,6

Prove that $\sim D$ from:

1.	$(A \bullet C) \rightarrow (B \vee C)$	Pr.
2.	$(B \vee C) \rightarrow \sim D$	Pr.
3.	$(A \bullet B) \bullet C$	Pr.
4.	$A \bullet B$	\bullet Elim 3
5.	C	\bullet Elim 3
6.	A	\bullet Elim 4
7.	$A \bullet C$	\bullet Int 6,5
8.	$B \vee C$	\rightarrow Elim 1, 7
9.	$\sim D$	\rightarrow Elim 2,8

Prove that D from:

1.	$(B \cdot A) \rightarrow D$	Pr.
2.	$(A \cdot C) \cdot B$	Pr.
3.	B	\bullet Elim 2
4.	$A \cdot C$	\bullet Elim 2
5.	A	\bullet Elim 4
6.	$B \cdot A$	\bullet Int 3,5
7.	D	\rightarrow Elim 1,6

Prove that $[(A \rightarrow B) \cdot (B \rightarrow C)] \cdot (A \rightarrow B)$ from:

1.	$A \rightarrow B$	Pr.
2.	$B \rightarrow C$	Pr.
3.	$(A \rightarrow B) \cdot (B \rightarrow C)$	\bullet Int 1,2
4.	$[(A \rightarrow B) \cdot (B \rightarrow C)] \cdot (A \rightarrow B)$	\bullet Int 3,1

Prove that A from:

1.	$B \rightarrow [B \rightarrow (B \rightarrow A)]$	Pr.
2.	B	Pr.
3.	$B \rightarrow (B \rightarrow A)$	\rightarrow Elim 1, 2
4.	$B \rightarrow A$	\rightarrow Elim 2,3
5.	A	\rightarrow Elim 2,4

Prove that D from:

1.	$(B \vee C) \rightarrow (A \cdot D)$	Pr.
2.	$(B \vee C) \cdot A$	Pr.
3.	$B \vee C$	\bullet Elim 2
4.	$A \cdot D$	\rightarrow Elim 1, 3
5.	D	\bullet Elim 4

Prove that $(C \cdot D) \cdot (A \vee B)$ from:

1.	$(A \vee B) \rightarrow C$	Pr.
2.	$D \rightarrow (A \vee B)$	Pr.
3.	D	Pr.
4.	$A \vee B$	\rightarrow Elim 2,3
5.	C	\rightarrow Elim 1,4
6.	$C \cdot D$	\bullet Int 3, 5
7.	$(C \cdot D) \cdot (A \vee B)$	\bullet Int 4, 6

Prove that $B \cdot C$ from:

1.	$A \rightarrow (A \rightarrow B)$	Pr.
2.	$(A \rightarrow B) \rightarrow C$	Pr.
3.	A	Pr.
4.	$A \rightarrow B$	\rightarrow Elim 1,3
5.	C	\rightarrow Elim 2,4
6.	B	\rightarrow Elim 3,4
7.	$B \cdot C$	\bullet Int 5, 6

Prove that $(A \cdot B) \cdot C$ from:

1.	$(A \vee B) \rightarrow C$	Pr.
2.	$A \rightarrow (A \rightarrow B)$	Pr.
3.	$(A \vee B) \cdot A$	Pr.
4.	$A \vee B$	\bullet Elim 3
5.	C	\rightarrow Elim 1,4
6.	A	\bullet Elim 3
7.	$A \rightarrow B$	\rightarrow Elim 2, 6
8.	B	\rightarrow Elim 6,7
9.	$A \cdot B$	\bullet Int 6, 8
10.	$(A \cdot B) \cdot C$	\bullet Int 9,5

Prove that $M \cdot N$ from:

1.	$(A \cdot B) \rightarrow [(L \rightarrow M) \rightarrow N]$	Pr.
2.	$A \cdot (L \rightarrow M)$	Pr.
3.	$L \cdot B$	Pr.
4.	A	\bullet Elim 2
5.	B	\bullet Elim 3
6.	$A \cdot B$	\bullet Int 4, 5
7.	$(L \rightarrow M) \rightarrow N$	\rightarrow Elim 1, 6
8.	$L \rightarrow M$	\bullet Elim 2
9.	N	\rightarrow Elim 7,8
10.	L	\bullet Elim 3
11.	M	\rightarrow Elim 8,10
12.	$M \cdot N$	\bullet Int 9, 11