Solutions to Workbook Exercises

Unit 9:

Substitution Instances

1.1. Proper Substitution Instances of Propositional Forms

Case [1b]

Proposition B → (A ∨ B) is a substitution instance of propositional form $p \rightarrow (q \vee p)$ since the proposition is created by means of the following substitution key:

p := B
q := A

$$\boxed{\text{B}}^{p} \rightarrow (\boxed{\text{A}}^{q} \vee \boxed{\text{B}}^{p})$$

Case [1c]

Proposition B → (C ∨ B) is a substitution instance of propositional form $p \rightarrow (q \vee p)$ since the proposition is created by means of the following substitution key:

p := B
q := C

$$\boxed{\text{B}}^{p} \rightarrow (\boxed{\text{C}}^{q} \vee \boxed{\text{B}}^{p})$$

Case [1d]

Proposition C → (A ∨ C) is a substitution instance of propositional form $p \rightarrow (q \vee p)$ since the proposition is created by means of the following substitution key:

p := C
q := A

$$\boxed{\text{C}}^{p} \rightarrow (\boxed{\text{A}}^{q} \vee \boxed{\text{C}}^{p})$$

Exercise “Substitution Key” – 1

(a)	$p \rightarrow (p \rightarrow q)$	$A \rightarrow (A \rightarrow B)$	$p :=$	A	$q :=$	B
(b)	$\sim p \rightarrow (p \bullet q)$	$\sim B \rightarrow (B \bullet C)$	$p :=$	B	$q :=$	C
(c)	$p \rightarrow (\sim q \rightarrow q)$	$A \rightarrow (\sim B \rightarrow B)$	$p :=$	A	$q :=$	B
(d)	$(p \bullet (\sim q \vee q)) \vee \sim p$	$(A \bullet (\sim C \vee C)) \vee \sim A$	$p :=$	A	$q :=$	C
(e)	$(p \rightarrow q) \equiv (\sim q \rightarrow \sim p)$	$(C \rightarrow B) \equiv (\sim B \rightarrow \sim C)$	$p :=$	C	$q :=$	B
(f)	$\sim(p \vee q) \equiv (\sim q \bullet \sim p)$	$\sim(B \vee A) \equiv (\sim A \bullet \sim B)$	$p :=$	B	$q :=$	A
(g)	$\sim q \vee (q \bullet \sim p)$	$\sim A \vee (A \bullet \sim C)$	$p :=$	C	$q :=$	A

2.2.When Propositions Are Not Substitution Instances of Propositional Forms

Example 1b

Explain why proposition $A \vee (B \rightarrow A)$ is a *not* a substitution instance of propositional form $p \rightarrow (q \vee p)$.
They have a different logical structure: proposition $A \vee (B \rightarrow A)$ is a disjunction while the propositional form $p \rightarrow (q \vee p)$ is a conditional.

Example 2b

Show that proposition $A \rightarrow (A \vee B)$ is a *not* a substitution instance of propositional form $p \rightarrow (q \vee p)$:

$p :=$ A / B
$q :=$ A

$$\boxed{\quad A \quad}^{p} \rightarrow (\boxed{\quad A \quad}^{q} \vee \boxed{\quad B \quad}^{p})$$

There is no consistent assignment to variable p.

Exercise “Proper Substitution Instances”

Note: When a proposition does not have the same logical form as the propositional form, then it is impossible to reconstruct a substitution key.

(a) $p \rightarrow (p \rightarrow q)$ $\quad$ $\mathrm{A} \rightarrow (\mathrm{B} \rightarrow \mathrm{A})$

☑ The proposition has the logical form of the propositional form.
There is a consistent substitution key: ☐ yes ☑ no
☐ The proposition does not have the logical form of the propositional form.

$p :=$ A / B
$q :=$ A

Therefore: The proposition
☐ is a substitution instance of the propositional form
☑ is not a substitution instance of the propositional form

(b) $p \rightarrow (p \rightarrow q)$ $\quad$ $(\mathrm{A} \rightarrow \mathrm{A}) \rightarrow \mathrm{B}$

☐ The proposition has the logical form of the propositional form.
There is a consistent substitution key: ☐ yes ☐ no
☑ The proposition does not have the logical form of the propositional form.

Therefore: The proposition
☐ is a substitution instance of the propositional form
☑ is not a substitution instance of the propositional form

(c) $p \rightarrow (p \rightarrow q)$ $\quad$ $\mathrm{A} \rightarrow \mathrm{B}$

☐ The proposition has the logical form of the propositional form.
There is a consistent substitution key: ☐ yes ☐ no
☑ The proposition does not have the logical form of the propositional form.

Therefore: The proposition
☐ is a substitution instance of the propositional form
☑ is not a substitution instance of the propositional form

(d) $p \rightarrow (p \bullet q)$ $\quad$ $\mathrm{A} \bullet (\mathrm{A} \rightarrow \mathrm{C})$

☐ The proposition has the logical form of the propositional form.
There is a consistent substitution key: ☐ yes ☐ no
☑ The proposition does not have the logical form of the propositional form.

Therefore: The proposition
☐ is a substitution instance of the propositional form
☑ is not a substitution instance of the propositional form

(e) $q \rightarrow (p \rightarrow q)$ $\quad$ $\mathrm{A} \rightarrow (\mathrm{B} \rightarrow \mathrm{B})$

☑ The proposition has the logical form of the propositional form.
There is a consistent substitution key: ☐ yes ☑ no
☐ The proposition does not have the logical form of the propositional form.

$p :=$ B
$q :=$ A / B

Therefore: The proposition
☐ is a substitution instance of the propositional form
☑ is not a substitution instance of the propositional form

(f) $q \rightarrow (p \rightarrow q)$ $A \rightarrow (B \rightarrow A)$

☑ The proposition has the logical form of the propositional form.
There is a consistent substitution key: ☑ yes ☐ no
☐ The proposition does not have the logical form of the propositional form.

p := B
q := A

Therefore: The proposition
☑ is a substitution instance of the propositional form
☐ is not a substitution instance of the propositional form

(g) $q \rightarrow (p \rightarrow q)$ $(A \rightarrow B) \rightarrow A$

☐ The proposition has the logical form of the propositional form.
There is a consistent substitution key: ☐ yes ☐ no
☑ The proposition does not have the logical form of the propositional form.

Therefore: The proposition
☐ is a substitution instance of the propositional form
☑ is not a substitution instance of the propositional form

(h) $q \rightarrow (p \rightarrow q)$ $A \vee (B \vee A)$

☐ The proposition has the logical form of the propositional form.
There is a consistent substitution key: ☐ yes ☐ no
☑ The proposition does not have the logical form of the propositional form.

Therefore: The proposition
☐ is a substitution instance of the propositional form
☑ is not a substitution instance of the propositional form

(i) $\sim p \vee q$ $\sim A \vee B$

☑ The proposition has the logical form of the propositional form.
There is a consistent substitution key: ☑ yes ☐ no
☐ The proposition does not have the logical form of the propositional form.

p := A
q := B

Therefore: The proposition
☑ is a substitution instance of the propositional form
☐ is not a substitution instance of the propositional form

(j) $\sim p \vee q$ $A \vee \sim B$

☐ The proposition has the logical form of the propositional form.
There is a consistent substitution key: ☐ yes ☐ no
☑ The proposition does not have the logical form of the propositional form.

Therefore: The proposition
☐ is a substitution instance of the propositional form
☑ is not a substitution instance of the propositional form

(k) $\sim p \vee q$ $\sim A \bullet B$

☐ The proposition has the logical form of the propositional form.
There is a consistent substitution key: ☐ yes ☐ no
☑ The proposition does not have the logical form of the propositional form.

Therefore: The proposition
☐ is a substitution instance of the propositional form
☐ is not a substitution instance of the propositional form

Exercise “Substitution Instances” – 1

$p := \sim A$
$q := B$
$r := C \lor D$
$s := \sim(A \rightarrow D)$

(a)

$p \lor q$

$\sim A \lor B$

(b)

$p \lor p$

$\sim A \lor \sim A$

(c)

$(p \rightarrow q) \lor q$

$(\sim A \rightarrow B) \lor B$

(d)

$q \rightarrow (p \lor q)$

$B \rightarrow (\sim A \lor B)$

(e)

$p \equiv (p \bullet q)$

$\sim A \equiv (\sim A \bullet B)$

(f)

$p \equiv (q \bullet p)$

$\sim A \equiv (B \bullet \sim A)$

(g)

$\sim(p \lor q)$

$\sim(\sim A \lor B)$

(h)

$\sim p \bullet \sim q$

$\sim\sim A \bullet \sim B$

(i)

$(\sim p \rightarrow \sim q) \lor q$

$(\sim\sim A \rightarrow \sim B) \lor B$

(j)

$(p \lor q) \bullet r$

$(\sim A \lor B) \bullet (C \lor D)$

(k)

$r \lor \sim r$

$(C \lor D) \lor \sim(C \lor D)$

(l)

$(\sim p \bullet \sim r) \rightarrow p$

$(\sim\sim A \bullet \sim(C \lor D)) \rightarrow \sim A$

(m)

$(p \bullet \sim p) \rightarrow s$

$(\sim A \bullet \sim\sim A) \rightarrow \sim(A \rightarrow D)$

(n)

$\sim s \lor q$

$\sim\sim(A \rightarrow D) \lor B$

(o)

$\sim(s \bullet q)$

$\sim(\sim(A \rightarrow D) \bullet B)$

(p)

$(\sim p \equiv \sim q) \lor (p \equiv \sim q)$

$(\sim\sim A \equiv \sim B) \lor (\sim A \equiv \sim B)$

(q)

$(p \rightarrow q) \rightarrow \sim(p \lor q)$

$(\sim A \rightarrow B) \rightarrow \sim(\sim A \lor B)$

(r)

$\sim(p \bullet q) \equiv (\sim p \lor \sim q)$

$\sim(\sim A \bullet B) \equiv (\sim\sim A \lor \sim B)$

(s)

$(\sim p \lor q) \equiv (p \rightarrow q)$

$(\sim\sim A \lor B) \equiv (\sim A \rightarrow B)$

Exercise “Substitution Instances” – 2

Construct substitution instances for each propositional form using four different substitution keys:

Key 1	**Key 2**	**Key 3**	**Key 4**
$p := \text{A}$	$p := \text{A}$	$p := {\sim}\text{C}$	$p := \text{A} \bullet \text{B}$
$q := \text{B}$	$q := \text{A}$	$q := {\sim}\text{D}$	$q := \text{B} \equiv \text{C}$

(a) $p \rightarrow q$

(1) $\text{A} \rightarrow \text{B}$

(2) $\text{A} \rightarrow \text{A}$

(3) ${\sim}\text{C} \rightarrow {\sim}\text{D}$

(4) $(\text{A} \bullet \text{B}) \rightarrow (\text{B} \equiv \text{C})$

(b) ${\sim}(p \rightarrow q) \rightarrow {\sim}q$

(1) ${\sim}(\text{A} \rightarrow \text{B}) \rightarrow {\sim}\text{B}$

(2) ${\sim}(\text{A} \rightarrow \text{A}) \rightarrow {\sim}\text{A}$

(3) ${\sim}({\sim}\text{C} \rightarrow {\sim}\text{D}) \rightarrow {\sim}{\sim}\text{D}$

(4) ${\sim}((\text{A} \bullet \text{B}) \rightarrow (\text{B} \equiv \text{C})) \rightarrow {\sim}(\text{B} \equiv \text{C})$

(c) $(p \vee q) \vee ({\sim}p \vee {\sim}q)$

(1) $(\text{A} \vee \text{B}) \vee ({\sim}\text{A} \vee {\sim}\text{B})$

(2) $(\text{A} \vee \text{A}) \vee ({\sim}\text{A} \vee {\sim}\text{A})$

(3) $({\sim}\text{C} \vee {\sim}\text{D}) \vee ({\sim}{\sim}\text{C} \vee {\sim}{\sim}\text{D})$

(4) $((\text{A} \bullet \text{B}) \vee (\text{B} \equiv \text{C})) \vee ({\sim}(\text{A} \bullet \text{B}) \vee {\sim}(\text{B} \equiv \text{C}))$

(d) $(p \bullet q) \vee (q \bullet p)$

(1) $(\text{A} \bullet \text{B}) \vee (\text{B} \bullet \text{A})$

(2) $(\text{A} \bullet \text{A}) \vee (\text{A} \bullet \text{A})$

(3) $({\sim}\text{C} \bullet {\sim}\text{D}) \vee ({\sim}\text{D} \bullet {\sim}\text{C})$

(4) $((\text{A} \bullet \text{B}) \bullet (\text{B} \equiv \text{C})) \vee ((\text{B} \equiv \text{C}) \bullet (\text{A} \bullet \text{B}))$

Key 1	**Key 2**	**Key 3**	**Key 4**
$p := A$	$p := A$	$p := \sim C$	$p := A \bullet B$
$q := B$	$q := A$	$q := \sim D$	$q := B \equiv C$

(e) $\sim(p \bullet q) \equiv (\sim p \vee \sim q)$

(1) $\sim(A \bullet B) \equiv (\sim A \vee \sim B)$

(2) $\sim(A \bullet A) \equiv (\sim A \vee \sim A)$

(3) $\sim(\sim C \bullet \sim D) \equiv (\sim\sim C \vee \sim\sim D)$

(4) $\sim((A \bullet B) \bullet (B \equiv C)) \equiv (\sim(A \bullet B) \vee \sim(B \equiv C))$

(f) $(p \vee q) \bullet (\sim p \vee \sim q)$

(1) $(A \vee B) \bullet (\sim A \vee \sim B)$

(2) $(A \vee A) \bullet (\sim A \vee \sim A)$

(3) $(\sim C \vee \sim D) \bullet (\sim\sim C \vee \sim\sim D)$

(4) $((A \bullet B) \vee (B \equiv C)) \bullet (\sim(A \bullet B) \vee \sim(B \equiv C))$

(g) $p \rightarrow [q \rightarrow (p \bullet q)]$

(1) $A \rightarrow [B \rightarrow (A \bullet B)]$

(2) $A \rightarrow [A \rightarrow (A \bullet A)]$

(3) $\sim C \rightarrow [\sim D \rightarrow (\sim C \bullet \sim D)]$

(4) $(A \bullet B) \rightarrow [(B \equiv C) \rightarrow ((A \bullet B) \bullet (B \equiv C))]$

(h) $[\sim p \rightarrow (p \rightarrow q)] \bullet \sim q$

(1) $[\sim A \rightarrow (A \rightarrow B)] \bullet \sim B$

(2) $[\sim A \rightarrow (A \rightarrow A)] \bullet \sim A$

(3) $[\sim\sim C \rightarrow (\sim C \rightarrow \sim D)] \bullet \sim\sim D$

(4) $[\sim(A \bullet B) \rightarrow ((A \bullet B) \rightarrow (B \equiv C))] \bullet \sim(B \equiv C)$

(i) $\sim[(\sim p \bullet q) \bullet p]$

(1) $\sim[(\sim A \bullet B) \bullet A]$

(2) $\sim[(\sim A \bullet A) \bullet A]$

(3) $\sim[(\sim\sim C \bullet \sim D) \bullet \sim C]$

(4) $\sim[(\sim(A \bullet B) \bullet (B \equiv C)) \bullet (A \bullet B)]$

Exercise “Substitution Key” – 2

(a)	$p \to (p \to q)$	$\mathrm{A} \to (\mathrm{A} \to \mathrm{A})$	$p := \mathrm{A}$ $q := \mathrm{A}$
(b)	$\sim p \to (p \bullet q)$	$\sim \mathrm{B} \to (\mathrm{B} \bullet \mathrm{C})$	$p := \mathrm{B}$ $q := \mathrm{C}$
(c)	$p \to (\sim q \to q)$	$\sim \mathrm{A} \to (\sim \mathrm{B} \to \mathrm{B})$	$p := \sim \mathrm{A}$ $q := \mathrm{B}$
(d)	$\sim p \to (p \to q)$	$\sim \mathrm{A} \to (\mathrm{A} \to \sim \mathrm{A})$	$p := \mathrm{A}$ $q := \sim \mathrm{A}$
(e)	$\sim p \to (p \to q)$	$\sim \mathrm{A} \to (\mathrm{A} \to (\mathrm{B} \lor \mathrm{C}))$	$p := \mathrm{A}$ $q := \mathrm{B} \lor \mathrm{C}$
(f)	$p \to (\sim q \to q)$	$\sim \mathrm{A} \to (\sim\sim \mathrm{B} \to \sim \mathrm{B})$	$p := \sim \mathrm{A}$ $q := \sim \mathrm{B}$
(g)	$(p \bullet (\sim q \lor q)) \lor \sim p$	$(\sim \mathrm{A} \bullet (\sim \mathrm{C} \lor \mathrm{C})) \lor \sim\sim \mathrm{A}$	$p := \sim \mathrm{A}$ $q := \mathrm{C}$
(h)	$(p \to q) \equiv (\sim q \to \sim p)$	$(\sim \mathrm{C} \to \sim \mathrm{B}) \equiv (\sim\sim \mathrm{B} \to \sim\sim \mathrm{C})$	$p := \sim \mathrm{C}$ $q := \sim \mathrm{B}$
(i)	$\sim(p \lor q) \equiv (\sim q \bullet \sim p)$	$\sim((\mathrm{A} \lor \mathrm{B}) \lor \mathrm{C}) \equiv (\sim \mathrm{C} \bullet \sim(\mathrm{A} \lor \mathrm{B}))$	$p := \mathrm{A} \lor \mathrm{B}$ $q := \mathrm{C}$
(j)	$(p \to q) \equiv (\sim q \to \sim p)$	$((\mathrm{A} \lor \mathrm{B}) \to \sim(\mathrm{B} \bullet \mathrm{A})) \equiv$ $\equiv (\sim\sim(\mathrm{B} \bullet \mathrm{A}) \to \sim(\mathrm{A} \lor \mathrm{B}))$	$p := \mathrm{A} \lor \mathrm{B}$ $q := \sim(\mathrm{B} \bullet \mathrm{A})$
(k)	$(p \bullet (\sim q \lor q)) \lor \sim p$	$(\mathrm{A} \bullet (\sim(\mathrm{B} \equiv \mathrm{D}) \lor (\mathrm{B} \equiv \mathrm{D}))) \lor \sim \mathrm{A}$	$p := \mathrm{A}$ $q := \mathrm{B} \equiv \mathrm{D}$
(l)	$\sim(p \lor q) \equiv (\sim q \bullet \sim p)$	$\sim((\mathrm{A} \lor \mathrm{B}) \lor (\sim \mathrm{B} \lor \mathrm{C})) \equiv$ $\equiv (\sim(\sim \mathrm{B} \lor \mathrm{C}) \bullet \sim(\mathrm{A} \lor \mathrm{B}))$	$p := \mathrm{A} \lor \mathrm{B}$ $q := \sim \mathrm{B} \lor \mathrm{C}$

Exercise “Substitution Instances” – 3

Propositions highlighted are substitutions instances of the propositional form in the middle:

(a)

1

A ≡ (A • B)

p := A
q := B

2

A ≡ (B • B)

p := A / B inconsistent
q := B

3

B ≡ (B • A)

p := B
q := A

4

A ≡ (A • A)

p := A
q := A

p ≡ (*p* • *q*)

5

~A ≡ (~A • C)

p := ~A
q := C

6

A ≡ (A • (C ∨ B))

p := A
q := C ∨ B

7

C • (~A ≡ ~A)

Different logical structure: This is not a biconditional!

9

~(A • B) ≡ (~(A • B) • D)

p := ~(A • B)
q := D

(b)

1

(~A ∨ ~B) • (A • B)

p := A
q := B

2

(~B ∨ ~A) • (B • A)

p := B
q := A

3

(~A ∨ ~A) • (A • A)

p := A
q := A

4

(~A ∨ ~B) • (~A • ~B)

p := A / ~A inconsistent
q := B / ~B inconsistent

(~*p* ∨ ~*q*) • (*p* • *q*)

5

(A ∨ B) • (~A • ~B)

Different logical structure: both disjuncts in the first parentheses of the proposition are not negations!

6

(~~B ∨ ~~B) • (~B • ~B)

p := ~B
q := ~B

7

(~A ∨ ~B) • (B • A)

p := A / B inconsistent
q := B / A inconsistent

9

(~~A ∨ ~~B) • (~A • ~B)

p := ~A
q := ~B

(c)

1

$\sim A \vee (B \rightarrow \sim A)$

Different logical structure: the proposition is not a conditional

2

$\sim B \rightarrow (\sim A \vee B)$

$p := A$
$q := B$

3

$\sim A \rightarrow (\sim A \vee B)$

$p := A$
$q :=$ A / B inconsist

4

$\sim\sim C \rightarrow (\sim(\sim C \rightarrow A) \vee \sim C)$

$p := \sim C \rightarrow A$
$q := \sim C$

$\boxed{\sim q \rightarrow (\sim p \vee q)}$

5

$\sim\sim(C \equiv \sim A) \rightarrow (\sim B \vee \sim(C \equiv \sim A))$

$p := B$
$q := \sim(C \equiv \sim A)$

6

$(A \vee B) \rightarrow (\sim C \vee \sim(A \vee B))$

Different logical structure: though the proposition is a conditional, its antecedent is not a negation.

7

$\sim(C \vee B) \rightarrow (\sim\sim B \vee (C \vee B))$

$p := \sim B$
$q := C \vee B$

9

$\sim C \rightarrow (\sim(A \vee B) \bullet C)$

Different logical structure: though the proposition is a conditional, its consequent is not a disjunction.

(d)

1

$\sim A \vee (B \bullet \sim A)$

Different logical structure: the proposition is not a conjunction.

2

$\sim B \bullet (\sim A \vee B)$

$p :=$ B / ~A inconsistent
$q := B$

3

$\sim A \bullet (\sim A \vee B)$

$p :=$ A / ~A inconsistent
$q := B$

4

$\sim\sim C \bullet (\sim(\sim C \rightarrow A) \vee \sim C)$

$p :=$ ~C / ~(~C → A) incons.
$q := \sim C$

$\boxed{\sim p \bullet (p \vee q)}$

5

$\sim\sim(C \vee \sim B) \bullet (\sim(C \vee \sim B) \vee B)$

$p := \sim(C \vee \sim B)$
$q := B$

6

$\sim(A \vee B) \bullet (\sim C \vee (A \vee B))$

$p :=$ A ∨ B / ~C inconsistent
$q := A \vee B$

7

$\sim(C \vee B) \bullet ((C \vee B) \vee \sim\sim\sim A)$

$p := C \vee B$
$q := \sim\sim\sim A$

9

$\sim C \bullet (C \vee C)$

$p := C$
$q := C$

Exercise “Substitution Instances” – 4

Key 1	**Key 2**
$p := A$	$p := {\sim}A$
$q := {\sim}B$	$q := {\sim}B$
$r := C \vee D$	$r := {\sim}B \bullet C$

Key 1 Key 2

(a)

$$\begin{array}{c} p \rightarrow q \\ p \\ \hline q \end{array}$$

$$\begin{array}{c} A \rightarrow {\sim}B \\ A \\ \hline {\sim}B \end{array} \qquad \begin{array}{c} {\sim}A \rightarrow {\sim}B \\ {\sim}A \\ \hline {\sim}B \end{array}$$

(b)

$$\begin{array}{c} p \rightarrow q \\ q \\ \hline p \end{array}$$

$$\begin{array}{c} A \rightarrow {\sim}B \\ {\sim}B \\ \hline A \end{array} \qquad \begin{array}{c} {\sim}A \rightarrow {\sim}B \\ {\sim}B \\ \hline {\sim}A \end{array}$$

(c)

$$\begin{array}{c} p \rightarrow q \\ {\sim}q \\ \hline {\sim}p \end{array}$$

$$\begin{array}{c} A \rightarrow {\sim}B \\ {\sim}{\sim}B \\ \hline {\sim}A \end{array} \qquad \begin{array}{c} {\sim}A \rightarrow {\sim}B \\ {\sim}{\sim}B \\ \hline {\sim}{\sim}A \end{array}$$

Key 1	**Key 2**
$p := \mathrm{A}$	$p := \sim\mathrm{A}$
$q := \sim\mathrm{B}$	$q := \sim\mathrm{B}$
$r := \mathrm{C} \vee \mathrm{D}$	$r := \sim\mathrm{B} \bullet \mathrm{C}$

Key 1 Key 2

(d)

$$\begin{array}{c} (p \bullet q) \rightarrow r \\ \sim r \\ \hline \sim p \vee \sim q \end{array}$$

$$\begin{array}{c} (\mathrm{A} \bullet \sim\mathrm{B}) \rightarrow (\mathrm{C} \vee \mathrm{D}) \\ \sim(\mathrm{C} \vee \mathrm{D}) \\ \hline \sim\mathrm{A} \vee \sim\sim\mathrm{B} \end{array}$$

$$\begin{array}{c} (\sim\mathrm{A} \bullet \sim\mathrm{B}) \rightarrow (\sim\mathrm{B} \bullet \mathrm{C}) \\ \sim(\sim\mathrm{B} \bullet \mathrm{C}) \\ \hline \sim\sim\mathrm{A} \vee \sim\sim\mathrm{B} \end{array}$$

(e)

$$\begin{array}{c} \sim(p \equiv q) \vee r \\ \sim r \bullet \sim(p \rightarrow q) \\ \hline \sim(q \rightarrow p) \end{array}$$

$$\begin{array}{c} \sim(\mathrm{A} \equiv \sim\mathrm{B}) \vee (\mathrm{C} \vee \mathrm{D}) \\ \sim(\mathrm{C} \vee \mathrm{D}) \bullet \sim(\mathrm{A} \rightarrow \sim\mathrm{B}) \\ \hline \sim(\sim\mathrm{B} \rightarrow \mathrm{A}) \end{array}$$

$$\begin{array}{c} \sim(\sim\mathrm{A} \equiv \sim\mathrm{B}) \vee (\sim\mathrm{B} \bullet \mathrm{C}) \\ \sim(\sim\mathrm{B} \bullet \mathrm{C}) \bullet \sim(\sim\mathrm{A} \rightarrow \sim\mathrm{B}) \\ \hline \sim(\sim\mathrm{B} \rightarrow \sim\mathrm{A}) \end{array}$$

(f)

$$\begin{array}{c} (\sim p \rightarrow q) \equiv r \\ r \bullet \sim q \\ \hline p \end{array}$$

$$\begin{array}{c} (\sim\mathrm{A} \rightarrow \sim\mathrm{B}) \equiv (\mathrm{C} \vee \mathrm{D}) \\ (\mathrm{C} \vee \mathrm{D}) \bullet \sim\sim\mathrm{B} \\ \hline \mathrm{A} \end{array}$$

$$\begin{array}{c} (\sim\sim\mathrm{A} \rightarrow \sim\mathrm{B}) \equiv (\sim\mathrm{B} \bullet \mathrm{C}) \\ (\sim\mathrm{B} \bullet \mathrm{C}) \bullet \sim\sim\mathrm{B} \\ \hline \sim\mathrm{A} \end{array}$$

Exercise “Substitution Key” – 3

(a)

$$\begin{array}{c} p \to q \\ \sim q \\ \hline \sim p \end{array} \qquad \begin{array}{c} A \to B \\ \sim B \\ \hline \sim A \end{array}$$

$p := A$

$q := B$

(b)

$$\begin{array}{c} p \vee \sim q \\ q \\ \hline p \end{array} \qquad \begin{array}{c} A \vee \sim A \\ A \\ \hline A \end{array}$$

$p := A$

$q := A$

(c)

$$\begin{array}{c} (p \equiv q) \vee p \\ p \bullet \sim q \\ \hline \sim p \end{array} \qquad \begin{array}{c} (\sim A \equiv \sim\sim C) \vee \sim A \\ \sim A \bullet \sim\sim\sim C \\ \hline \sim\sim A \end{array}$$

$p := \sim A$

$q := \sim\sim C$

(d)

$$\begin{array}{c} (p \to q) \bullet q \\ p \vee \sim q \\ \hline \sim p \end{array} \qquad \begin{array}{c} (\sim C \to (A \vee B)) \bullet (A \vee B) \\ \sim C \vee \sim(A \vee B) \\ \hline \sim\sim C \end{array}$$

$p := \sim C$

$q := A \vee B$

(e)

$$\begin{array}{c} (p \vee q) \bullet (q \vee p) \\ \sim q \\ \hline p \end{array} \qquad \begin{array}{c} (\sim B \vee A) \bullet (A \vee \sim B) \\ \sim A \\ \hline \sim B \end{array}$$

$p := \sim B$

$q := A$

(f)

$$\begin{array}{c} (p \to q) \bullet (q \to p) \\ q \\ \hline p \end{array} \qquad \begin{array}{c} (\sim(B \bullet C) \to A) \bullet (A \to \sim(B \bullet C)) \\ A \\ \hline \sim(B \bullet C) \end{array}$$

$p := \sim(B \bullet C)$

$q := A$

(g)

$$\begin{array}{c} (p \to q) \bullet (q \to p) \\ \sim q \\ \hline \sim p \end{array} \qquad \begin{array}{c} ((C \vee D) \to B) \bullet (B \to (C \vee D)) \\ \sim B \\ \hline \sim(C \vee D) \end{array}$$

$p := C \vee D$

$q := B$

(h)

$$\begin{array}{c} (p \vee q) \bullet (q \to p) \\ \sim q \\ \hline p \end{array} \qquad \begin{array}{c} (A \vee (\sim B \equiv C)) \bullet ((\sim B \equiv C) \to A) \\ \sim(\sim B \equiv C) \\ \hline A \end{array}$$

$p := A$

$q := \sim B \equiv C$

Exercise “Substitution Instances” – 5

(a)

1

$$\begin{array}{c} A \equiv B \\ \sim A \\ \hline \sim B \end{array}$$

$p := A$
$q := B$

2

$$\begin{array}{c} A \equiv B \\ \sim B \\ \hline \sim A \end{array}$$

$p := A$ / B inconsistent
$q := B$ / A inconsistent

3

$$\begin{array}{c} C \equiv B \\ \sim C \\ \hline \sim B \end{array}$$

$p := C$
$q := B$

4

$$\begin{array}{c} \sim A \equiv \sim B \\ \sim A \\ \hline \sim B \end{array}$$

$p := \sim A$ / A inconsistent
$q := \sim B$ / B inconsistent

$$\boxed{\begin{array}{c} p \equiv q \\ \sim p \\ \hline \sim q \end{array}}$$

5

$$\begin{array}{c} \sim C \equiv B \\ \sim C \\ \hline \sim B \end{array}$$

$p := \sim C$ / C inconsistent
$q := B$

6

$$\begin{array}{c} (A \lor B) \equiv (C \bullet D) \\ \sim(A \lor B) \\ \hline \sim(C \bullet D) \end{array}$$

$p := A \lor B$
$q := C \bullet D$

7

$$\begin{array}{c} (A \lor B) \equiv (C \bullet D) \\ \sim A \lor B \\ \hline \sim C \bullet D \end{array}$$

Different logical structure: the second premise is not a negation; nor is the conclusion is a negation!

9

$$\begin{array}{c} \sim(A \lor B) \equiv (\sim C \bullet D) \\ \sim\sim(A \lor B) \\ \hline \sim(C \bullet D) \end{array}$$

$p := \sim(A \lor B)$
$q := \sim C \bullet D$ / $C \bullet D$ inconsistent

(b)

1

$$\begin{array}{c} A \rightarrow B \\ \sim A \lor \sim B \\ \hline \sim(\sim A \bullet \sim B) \end{array}$$

$p := A$ / ~A inconsistent
$q := B$ / ~B inconsistent

2

$$\begin{array}{c} C \rightarrow D \\ \sim C \lor \sim D \\ \hline \sim(C \bullet D) \end{array}$$

$p := C$
$q := D$

3

$$\begin{array}{c} A \rightarrow C \\ \sim(A \bullet C) \\ \hline \sim A \lor \sim C \end{array}$$

Different logical structure: the conclusion is not a negation; the second premise is not a disjunction!

4

$$\begin{array}{c} \sim A \rightarrow \sim B \\ \sim\sim A \lor \sim\sim B \\ \hline \sim(\sim A \bullet \sim B) \end{array}$$

$p := \sim A$
$q := \sim B$

$$\boxed{\begin{array}{c} p \rightarrow q \\ \sim p \lor \sim q \\ \hline \sim(p \bullet q) \end{array}}$$

5

$$\begin{array}{c} \sim A \rightarrow \sim A \\ \sim\sim A \lor \sim\sim A \\ \hline \sim(\sim A \bullet \sim A) \end{array}$$

$p := \sim A$
$q := \sim A$

6

$$\begin{array}{c} (A \bullet B) \equiv \sim C \\ \sim(A \bullet B) \lor \sim\sim C \\ \hline \sim((A \bullet B) \bullet \sim C) \end{array}$$

Different logical structure: the first premise is not a conditional!

7

$$\begin{array}{c} (A \lor C) \rightarrow \sim(A \equiv B) \\ \sim(A \lor C) \lor \sim\sim(A \equiv B) \\ \hline \sim((A \lor C) \bullet \sim(A \equiv B)) \end{array}$$

$p := A \lor C$
$q := \sim(A \equiv B)$

9

$$\begin{array}{c} A \rightarrow B \\ A \lor B \\ \hline \sim(\sim A \bullet \sim B) \end{array}$$

Different logical structure: the disjuncts in the second premise are not negations!