Workbook Unit 9:

Substitution Instances

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Overview

We have already worked with some intuitive understanding of the distinction between a proposition, which is a proper substitution instance of a propositional form. In this unit, we will introduce the notion of a substitution instance more systematically. Your grasping of this idea is crucial in preparation for the Proofs Units.

This unit

- reminds you of the idea of proper substitution instance
- introduces the notion of a substitution instance

Prerequisites

There are no prerequisites for this unit but it would be helpful for you to remind yourself of the idea of a proper substitution instance (see Unit 6).

1. Propositions and Propositional Forms

1.1. Examples

Consider the group of propositions:

(1) If Susan gets either 99 or 100 points on her quizzes, she will get an A+.
(2) If it rains or snows, it is good to take an umbrella.
(3) If $x = 3$ or $x = 4$ then $x < 5$.

They are all different but they share something. They share a *logical form*, which can be represented thus:

$$(p \vee q) \rightarrow r$$

Consider another group of propositions:

(4) Either it rains or it does not rain
(5) Either the food is good or it is not good
(6) I should die or I should not die

Again, these are different propositions that share the form:

$$p \vee {\sim}p$$

1.2. Propositional Forms and Propositional Variables

You already know that propositional forms capture the logical structure of propositions. Unlike propositions, propositional forms do not *say* anything – this is because propositional forms are formulated by means of propositional variables.

> A propositional variable is simply a letter for which, or in place of which, a (simple or compound) proposition may be substituted.

In our ordinary language, variables do not occur. When we try to capture the structure of what somebody said we would use senseless words like 'duh', or 'blah'. It is important, however, to distinguish sharply between propositional variables and propositional constants. Propositional constants just abbreviate propositions. Propositional variables do not abbreviate anything – they are places in which propositional constants can jump – this is why it is good to think about variables as boxes.

If you really deeply grasp the difference between a propositional variable and a propositional constant, you are half way there to grasping the difficult idea of a substitution instance.

Superficially (this is a reminder), the difference between propositional variables and propositional constants is marked by the symbols we use for both. Capital letters (A, B, C, …) are used for propositional constants. Small letters from the middle of the alphabet (*p*, *q*, *r*, *s*, …) are used for propositional variables.

2. Substitution Instances of Propositional Forms

We will introduce the notion of a substitution instance of a propositional form (§2.3) but we will begin with a special case, viz. the notion of a proper substitution instance (§2.1), which will be a kind of reminder of the notion we introduced in Unit 6.

2.1. Proper Substitution Instances of Propositional Forms

We have already introduced the notion of a proper substitution instance of a propositional form:

> A proper substitution instance of a propositional form $\mathcal{F}$ is any proposition $\mathcal{P}$, which is the result of replacing propositional variables in $\mathcal{F}$ by simple propositions, where all same-shaped variables are replaced by the same proposition, and all differently shaped variables are replaced by a different proposition.

Take, for example, the propositional form:

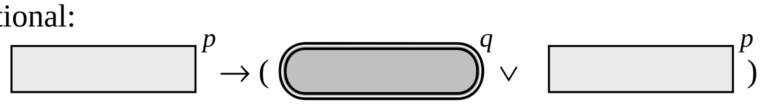
{1} $p \rightarrow (q \vee p)$

There are two propositional variables in it: *p* and *q*. Here are some proper substitution instances of the proposition form {1}:

[1a] $\mathrm{A} \rightarrow (\mathrm{B} \vee \mathrm{A})$
[1b] $\mathrm{B} \rightarrow (\mathrm{A} \vee \mathrm{B})$
[1c] $\mathrm{B} \rightarrow (\mathrm{C} \vee \mathrm{B})$
[1d] $\mathrm{C} \rightarrow (\mathrm{A} \vee \mathrm{C})$

We can think of a "**substitution key**", in each of the cases, which provides us with the information what propositional constant was substituted for what propositional variable.

We can think about this propositional form as having two kinds of boxes, the first of which occurs twice: in the antecedent of the main conditional and as the second disjunct in the disjunction, which occurs in the consequent of the main conditional:

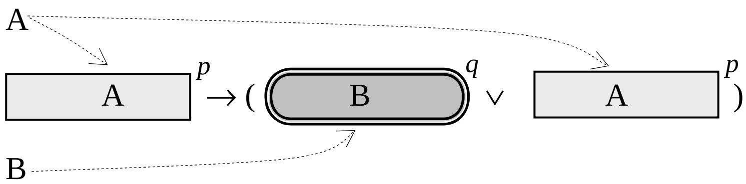


Case [1a]

In case [1a] the substitution key can be represented thus:

$p := \mathrm{A}$
$q := \mathrm{B}$

Note that the expression '$p := \mathrm{A}$' means that proposition A is to be substituted for all occurrences of the propositional variable *p*. Graphically:



Proposition A → (B ∨ A) is a (proper) substitution instance of $p \to (q \lor p)$.

Explain why the remaining cases are substitution instances of the propositional form $p \to (q \lor p)$ by constructing appropriate substitution keys:

Case [1b]

Proposition B → (A ∨ B) is a substitution instance of propositional form $p \to (q \lor p)$ since the proposition is created by means of the following substitution key:

p :=
q :=

Case [1c]

Proposition B → (C ∨ B) is a substitution instance of propositional form $p \to (q \lor p)$ since the proposition is created by means of the following substitution key:

p :=
q :=

Case [1d]

Proposition C → (A ∨ C) is a substitution instance of propositional form $p \to (q \lor p)$ since the proposition is created by means of the following substitution key:

p :=
q :=

Find some other proper substitution instances of the propositional form $p \to (q \lor p)$:

p :=
q :=

p :=
q :=

Exercise "Substitution Key" – 1

You are given a propositional form and its proper substitution instance. Your task is to reconstruct the substitution key used to create the substitution instance:

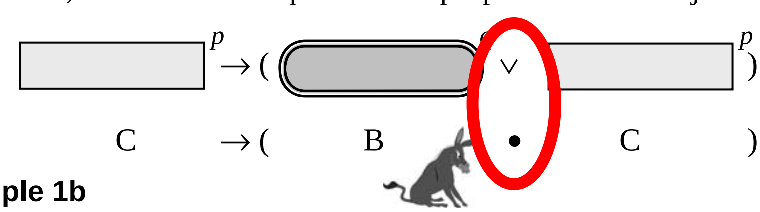
(a)	$p \rightarrow (p \rightarrow q)$	A → (A → B)	p :=		q :=	
(b)	$\sim p \rightarrow (p \bullet q)$	~B → (B • C)	p :=		q :=	
(c)	$p \rightarrow (\sim q \rightarrow q)$	A → (~B → B)	p :=		q :=	
(d)	$(p \bullet (\sim q \vee q)) \vee \sim p$	(A • (~C ∨ C)) ∨ ~A	p :=		q :=	
(e)	$(p \rightarrow q) \equiv (\sim q \rightarrow \sim p)$	(C → B) ≡ (~B → ~C)	p :=		q :=	
(f)	$\sim(p \vee q) \equiv (\sim q \bullet \sim p)$	~(B ∨ A) ≡ (~A • ~B)	p :=		q :=	
(g)	$\sim q \vee (q \bullet \sim p)$	~A ∨ (A • ~C)	p :=		q :=	

2.2. When Propositions Are Not Substitution Instances of Propositional Forms

The following two examples illustrate the two ways in which a proposition may fail to be a substitution instance of a propositional form. First, a proposition may fail to have the same logical structure as the propositional form. Second, a proposition may have the same logical structure as the propositional form but it may still be impossible to find a consistent substitution key by means of which to construct the proposition from the propositional form.

Example 1a

Is proposition C → (B • C) a substitution instance of propositional form $p \rightarrow (q \vee p)$? Here the answer is immediately negative. The logical structure of proposition C → (B • C) differs from the logical structure of propositional form $p \rightarrow (q \vee p)$ – you cannot obtain proposition C → (B • C) from propositional form $p \rightarrow (q \vee p)$ by substituting the propositions C and B, respectively, for the propositional variables p and q, respectively. This is because the consequent in the propositional form is a disjunction, while the consequent in the proposition is a conjunction.



Example 1b

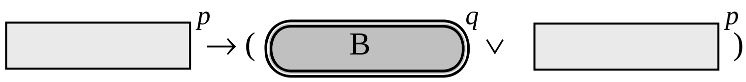
Explain why proposition A ∨ (B → A) is a *not* a substitution instance of propositional form $p \rightarrow (q \vee p)$:

Example 2a

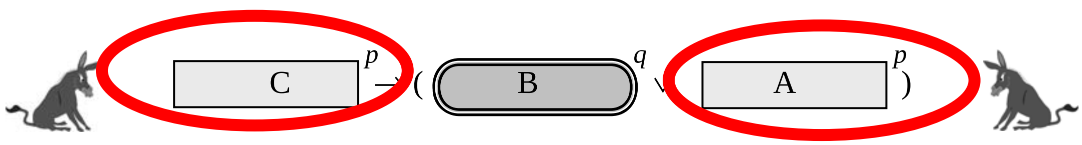
Is proposition C → (B ∨ A) a substitution instance of propositional form $p \rightarrow (q \vee p)$?

p := C / A inconsistent!
q := C

When you try to reconstruct the substitution key, you will encounter a problem. There is no problem in finding the assignment for variable q: it is substituted by the proposition B.



However, there is no consistent assignment for variable p. All the occurrences of the propositional variables need to be substituted uniformly. Here two different propositions (C and A) would purport to be substituting for p – that is forbidden!



If we were to reconstruct the substitution key, it would have a consistent assignment for variable q (q := B), but an inconsistent assignment for variable p (p := C, A).

You can see this intuitively when you try to formulate the propositional form $p \rightarrow (q \vee p)$ in words. It is a conditional whose consequent is a disjunction, where the second disjunct is just the same proposition as that contained in the antecedent of the conditional. But while C → (B ∨ A) is a conditional whose consequent is a disjunction, the second disjunct (viz. A) is *not* the same proposition as that contained in the antecedent of the conditional (viz. C).

The possibility of finding a *consistent* substitution key provides an operational criterion for deciding whether a proposition is a substitution instance of a given propositional form. If it is possible to find a substitution key by means of which a proposition was constructed from a propositional form, then the proposition is a substitution instance of that propositional form (see p. 9-11 for a more precise definition).

Example 2b

Show that proposition A → (A ∨ B) is a *not* a substitution instance of propositional form $p \rightarrow (q \vee p)$:

p :=
q :=

Exercise "Proper Substitution Instances"

You are given a propositional form and you are to decide whether the proposition next to it is its (proper) substitution instance.

(a) $p \rightarrow (p \rightarrow q)$ A → (B → A)

☐ The proposition has the logical form of the propositional form.
There is a consistent substitution key: ☐ yes ☐ no
☐ The proposition does not have the logical form of the propositional form.

$p :=$
$q :=$

Therefore: The proposition
☐ is a substitution instance of the propositional form
☐ is not a substitution instance of the propositional form

(b) $p \rightarrow (p \rightarrow q)$ (A → A) → B

☐ The proposition has the logical form of the propositional form.
There is a consistent substitution key: ☐ yes ☐ no
☐ The proposition does not have the logical form of the propositional form.

$p :=$
$q :=$

Therefore: The proposition
☐ is a substitution instance of the propositional form
☐ is not a substitution instance of the propositional form

(c) $p \rightarrow (p \rightarrow q)$ A → B

☐ The proposition has the logical form of the propositional form.
There is a consistent substitution key: ☐ yes ☐ no
☐ The proposition does not have the logical form of the propositional form.

$p :=$
$q :=$

Therefore: The proposition
☐ is a substitution instance of the propositional form
☐ is not a substitution instance of the propositional form

(d) $p \rightarrow (p \bullet q)$ A • (A → C)

☐ The proposition has the logical form of the propositional form.
There is a consistent substitution key: ☐ yes ☐ no
☐ The proposition does not have the logical form of the propositional form.

$p :=$
$q :=$

Therefore: The proposition
☐ is a substitution instance of the propositional form
☐ is not a substitution instance of the propositional form

(e) $q \rightarrow (p \rightarrow q)$ A → (B → B)

☐ The proposition has the logical form of the propositional form.
There is a consistent substitution key: ☐ yes ☐ no
☐ The proposition does not have the logical form of the propositional form.

$p :=$
$q :=$

Therefore: The proposition
☐ is a substitution instance of the propositional form
☐ is not a substitution instance of the propositional form

(f) $q \to (p \to q)$ A → (B → A)

☐ The proposition has the logical form of the propositional form.
There is a consistent substitution key: ☐ yes ☐ no
☐ The proposition does not have the logical form of the propositional form.

p :=
q :=

Therefore: The proposition
☐ is a substitution instance of the propositional form
☐ is not a substitution instance of the propositional form

(g) $q \to (p \to q)$ (A → B) → A

☐ The proposition has the logical form of the propositional form.
There is a consistent substitution key: ☐ yes ☐ no
☐ The proposition does not have the logical form of the propositional form.

p :=
q :=

Therefore: The proposition
☐ is a substitution instance of the propositional form
☐ is not a substitution instance of the propositional form

(h) $q \to (p \to q)$ A ∨ (B ∨ A)

☐ The proposition has the logical form of the propositional form.
There is a consistent substitution key: ☐ yes ☐ no
☐ The proposition does not have the logical form of the propositional form.

p :=
q :=

Therefore: The proposition
☐ is a substitution instance of the propositional form
☐ is not a substitution instance of the propositional form

(i) $\sim p \lor q$ ~A ∨ B

☐ The proposition has the logical form of the propositional form.
There is a consistent substitution key: ☐ yes ☐ no
☐ The proposition does not have the logical form of the propositional form.

p :=
q :=

Therefore: The proposition
☐ is a substitution instance of the propositional form
☐ is not a substitution instance of the propositional form

(j) $\sim p \lor q$ A ∨ ~B

☐ The proposition has the logical form of the propositional form.
There is a consistent substitution key: ☐ yes ☐ no
☐ The proposition does not have the logical form of the propositional form.

p :=
q :=

Therefore: The proposition
☐ is a substitution instance of the propositional form
☐ is not a substitution instance of the propositional form

(k) $\sim p \lor q$ ~A • B

☐ The proposition has the logical form of the propositional form.
There is a consistent substitution key: ☐ yes ☐ no
☐ The proposition does not have the logical form of the propositional form.

p :=
q :=

Therefore: The proposition
☐ is a substitution instance of the propositional form
☐ is not a substitution instance of the propositional form

2.3. Substitution Instances of Propositional Forms

The notion of a proper substitution instance is a special case of the more general notion of a substitution instance.

> A substitution instance of a propositional form $\mathcal{F}$ is any proposition $\mathcal{P}$, which is the result of replacing propositional variables in $\mathcal{F}$ by some propositions (simple or complex), where all same-shaped variables are replaced by the same proposition.

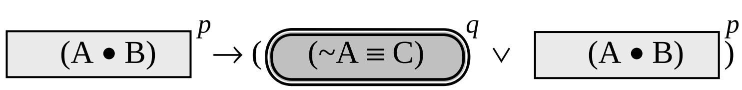
The definition of a substitution instance is thus less restrictive than the definition of a proper substitution instance. There are two main differences. First, whereas proper substitution instances are the result of replacing propositional variables by *simple* propositions, substitution instances are the result of replacing propositional variables by any – simple or complex – propositions. Second, whereas in the case of a proper substitution instances it is required that differently shaped variables be replaced by different propositions, this requirement is absent in the case of substitution instances.

Let's see on two examples what the lack of these restrictions means. Let's still work with the propositional form $p \rightarrow (q \vee p)$.

Example 1: Complex propositions

Complex propositions can be substituted for propositional variables:

p := A • B
q := ~A ≡ C

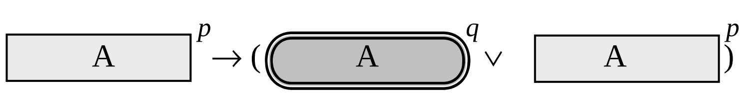


Proposition (A • B) → ((~A ≡ C) ∨ (A • B)) is a substitution instance of $p \rightarrow (q \vee p)$.

Example 2: Same propositions substituted for differently shaped variables

The same proposition can be substituted for differently shaped variables:

p := A
q := A



Proposition A → (A ∨ A) is a substitution instance of $p \rightarrow (q \vee p)$.

You remember that you can characterize the propositional form as a conditional whose consequent is a disjunction, where the second disjunct is just the same proposition as that contained in the antecedent of the conditional. The proposition A → (A ∨ A) is indeed a conditional whose consequent is a disjunction, where the second disjunct (viz. A) is just the same proposition as that contained in the antecedent of the conditional (viz. A). Intuitively, there are even more restrictions here but this does not in any way affect the fact that A → (A ∨ A) fulfills the conditions characteristic of the form $p \rightarrow (q \vee p)$. In a slogan: a substitution instance can do more but never less.

We can now formulate an operational criterion for deciding whether a proposition is a substitution instance of a given propositional form and whether it is a proper substitution instance of that form:

A proposition $\mathcal{P}$ is a **substitution instance** of a propositional form $\mathcal{F}$ iff there is a consistent substitution key, which can be used to construct proposition $\mathcal{P}$ from proposition form $\mathcal{F}$.
A proposition $\mathcal{P}$ is a **proper substitution instance** of a propositional form $\mathcal{F}$ iff there is a consistent substitution key, which can be used to construct proposition $\mathcal{P}$ from proposition form $\mathcal{F}$ and the substitution key fulfills two conditions: first, only simple propositions are assigned to variables and , second, different propositions are assigned to different variables.

Exercise "Substitution Instances" – 1

Construct the substitution instances of the listed propositional forms using the following substitution key:

p := ~A
q := B
r := C ∨ D
s := ~(A → D)

(a) $p \vee q$

(b) $p \vee p$

(c) $(p \rightarrow q) \vee q$

(d) $q \rightarrow (p \vee q)$

(e) $p \equiv (p \bullet q)$

(f) $p \equiv (q \bullet p)$

(g) $\sim(p \vee q)$

(h) $\sim p \bullet \sim q$

(i) $(\sim p \rightarrow \sim q) \vee q$

(j) $(p \vee q) \bullet r$

(k) $r \vee \sim r$

(l) $(\sim p \bullet \sim r) \rightarrow p$

(m) $(p \bullet \sim p) \rightarrow s$

(n) $\sim s \vee q$

(o) $\sim(s \bullet q)$

(p) $(\sim p \equiv \sim q) \vee (p \equiv \sim q)$

(q) $(p \rightarrow q) \rightarrow \sim(p \vee q)$

(r) $\sim(p \bullet q) \equiv (\sim p \vee \sim q)$

(s) $(\sim p \vee q) \equiv (p \rightarrow q)$

Exercise "Substitution Instances" – 2

Construct substitution instances for each propositional form using four different substitution keys:

Key 1	Key 2	Key 3	Key 4
p := A	p := A	p := ~C	p := A • B
q := B	q := A	q := ~D	q := B ≡ C

(a) $p \rightarrow q$

(1)

(2)

(3)

(4)

(b) $\sim(p \rightarrow q) \rightarrow \sim q$

(1)

(2)

(3)

(4)

(c) $(p \vee q) \vee (\sim p \vee \sim q)$

(1)

(2)

(3)

(4)

(d) $(p \bullet q) \vee (q \bullet p)$

(1)

(2)

(3)

(4)

	Key 1	**Key 2**	**Key 3**	**Key 4**
	p := A	p := A	p := ~C	p := A • B
	q := B	q := A	q := ~D	q := B ≡ C

(e) ~(p • q) ≡ (~p ∨ ~q)

(1)

(2)

(3)

(4)

(f) (p ∨ q) • (~p ∨ ~q)

(1)

(2)

(3)

(4)

(g) p → [q → (p • q)]

(1)

(2)

(3)

(4)

(h) [~p → (p → q)] • ~q

(1)

(2)

(3)

(4)

(i) ~[(~p • q) • p]

(1)

(2)

(3)

(4)

Exercise "Substitution Key" – 2

You are given a propositional form and its substitution instance. Your task is to reconstruct the substitution key used to create the substitution instance:

	Form	Instance	Key
(a)	$p \rightarrow (p \rightarrow q)$	$A \rightarrow (A \rightarrow A)$	$p :=$ $q :=$
(b)	$\sim p \rightarrow (p \bullet q)$	$\sim B \rightarrow (B \bullet C)$	$p :=$ $q :=$
(c)	$p \rightarrow (\sim q \rightarrow q)$	$\sim A \rightarrow (\sim B \rightarrow B)$	$p :=$ $q :=$
(d)	$\sim p \rightarrow (p \rightarrow q)$	$\sim A \rightarrow (A \rightarrow \sim A)$	$p :=$ $q :=$
(e)	$\sim p \rightarrow (p \rightarrow q)$	$\sim A \rightarrow (A \rightarrow (B \vee C))$	$p :=$ $q :=$
(f)	$p \rightarrow (\sim q \rightarrow q)$	$\sim A \rightarrow (\sim\sim B \rightarrow \sim B)$	$p :=$ $q :=$
(g)	$(p \bullet (\sim q \vee q)) \vee \sim p$	$(\sim A \bullet (\sim C \vee C)) \vee \sim\sim A$	$p :=$ $q :=$
(h)	$(p \rightarrow q) \equiv (\sim q \rightarrow \sim p)$	$(\sim C \rightarrow \sim B) \equiv (\sim\sim B \rightarrow \sim\sim C)$	$p :=$ $q :=$
(i)	$\sim(p \vee q) \equiv (\sim q \bullet \sim p)$	$\sim((A \vee B) \vee C) \equiv (\sim C \bullet \sim(A \vee B))$	$p :=$ $q :=$
(j)	$(p \rightarrow q) \equiv (\sim q \rightarrow \sim p)$	$((A \vee B) \rightarrow \sim(B \bullet A)) \equiv$ $\equiv (\sim\sim(B \bullet A) \rightarrow \sim(A \vee B))$	$p :=$ $q :=$
(k)	$(p \bullet (\sim q \vee q)) \vee \sim p$	$(A \bullet (\sim(B \equiv D) \vee (B \equiv D))) \vee \sim A$	$p :=$ $q :=$
(l)	$\sim(p \vee q) \equiv (\sim q \bullet \sim p)$	$\sim((A \vee B) \vee (\sim B \vee C)) \equiv$ $\equiv (\sim(\sim B \vee C) \bullet \sim(A \vee B))$	$p :=$ $q :=$

Exercise "Substitution Instances" – 3

You are given a propositional form in the center and nine potential substitution instances around it. Your task is to say which of the propositions are substitution instances of the propositional form. For each of the substitution instances, reconstruct the substitution key used to create it:

(a)

Central form: $p \equiv (p \bullet q)$

1. $A \equiv (A \bullet B)$
   $p :=$
   $q :=$

2. $A \equiv (B \bullet B)$
   $p :=$
   $q :=$

3. $B \equiv (B \bullet A)$
   $p :=$
   $q :=$

4. $A \equiv (A \bullet A)$
   $p :=$
   $q :=$

5. $\sim A \equiv (\sim A \bullet C)$
   $p :=$
   $q :=$

6. $A \equiv (A \bullet (C \vee B))$
   $p :=$
   $q :=$

7. $C \bullet (\sim A \equiv \sim A)$
   $p :=$
   $q :=$

9. $\sim(A \bullet B) \equiv (\sim(A \bullet B) \bullet D)$
   $p :=$
   $q :=$

(b)

Central form: $(\sim p \vee \sim q) \bullet (p \bullet q)$

1. $(\sim A \vee \sim B) \bullet (A \bullet B)$
   $p :=$
   $q :=$

2. $(\sim B \vee \sim A) \bullet (B \bullet A)$
   $p :=$
   $q :=$

3. $(\sim A \vee \sim A) \bullet (A \bullet A)$
   $p :=$
   $q :=$

4. $(\sim A \vee \sim B) \bullet (\sim A \bullet \sim B)$
   $p :=$
   $q :=$

5. $(A \vee B) \bullet (\sim A \bullet \sim B)$
   $p :=$
   $q :=$

6. $(\sim\sim B \vee \sim\sim B) \bullet (\sim B \bullet \sim B)$
   $p :=$
   $q :=$

7. $(\sim A \vee \sim B) \bullet (B \bullet A)$
   $p :=$
   $q :=$

9. $(\sim\sim A \vee \sim\sim B) \bullet (\sim A \bullet \sim B)$
   $p :=$
   $q :=$

(c)

1	2	3
$\sim A \lor (B \rightarrow \sim A)$ $p :=$ $q :=$	$\sim B \rightarrow (\sim A \lor B)$ $p :=$ $q :=$	$\sim A \rightarrow (\sim A \lor B)$ $p :=$ $q :=$
4		**5**
$\sim\sim C \rightarrow (\sim(\sim C \rightarrow A) \lor \sim C)$ $p :=$ $q :=$	$\sim q \rightarrow (\sim p \lor q)$	$\sim\sim(C \equiv \sim A) \rightarrow (\sim B \lor \sim(C \equiv \sim A))$ $p :=$ $q :=$
6	**7**	**9**
$(A \lor B) \rightarrow (\sim C \lor \sim(A \lor B))$ $p :=$ $q :=$	$\sim(C \lor B) \rightarrow (\sim\sim B \lor (C \lor B))$ $p :=$ $q :=$	$\sim C \rightarrow (\sim(A \lor B) \bullet C)$ $p :=$ $q :=$

(d)

1	2	3
$\sim A \lor (B \bullet \sim A)$ $p :=$ $q :=$	$\sim B \bullet (\sim A \lor B)$ $p :=$ $q :=$	$\sim A \bullet (\sim A \lor B)$ $p :=$ $q :=$
4		**5**
$\sim\sim C \bullet (\sim(\sim C \rightarrow A) \lor \sim C)$ $p :=$ $q :=$	$\sim p \bullet (p \lor q)$	$\sim\sim(C \lor \sim B) \bullet (\sim(C \lor \sim B) \lor B)$ $p :=$ $q :=$
6	**7**	**9**
$\sim(A \lor B) \bullet (\sim C \lor (A \lor B))$ $p :=$ $q :=$	$\sim(C \lor B) \bullet ((C \lor B) \lor \sim\sim\sim A)$ $p :=$ $q :=$	$\sim C \bullet (C \lor C)$ $p :=$ $q :=$

You can also do the exercise Ex.09. Make Substitution (1)-(4) on-line.

You can also do the exercise Ex.09. Find Substitution (1)-(2) on-line.

Note that you will learn most effectively when you do those exercises in the order in which they are listed (their difficulty increases). Review the “quiz results” each time to understand your mistakes. When you score 100% on them you are ready to go on to the next exercise.

3. Arguments and Argument Forms

So far we have been talking about substitution instances of propositional forms. Substitution instances of propositional forms are propositions. We will now extend the notion to cover substitution instances of argument forms. Substitution instances of argument forms are arguments.

3.1. Argument Forms

Consider the two arguments:

(1) If Susan gets either 99 or 100 points on her quizzes, she will get an A+.
Susan did not get an A+.
So Susan got neither 99 nor 100 points on her quizzes.

(2) If it rains or snows, it is good to take an umbrella.
It was silly of me to have taken the umbrella.
So, it neither rained nor snowed.

These two arguments share a *logical form*, which can be represented thus:

$$\begin{array}{c} (p \vee q) \rightarrow r \\ \sim r \\ \hline \sim p \bullet \sim q \end{array}$$

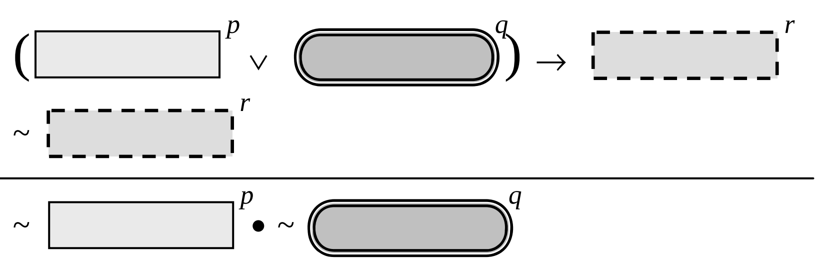
3.2. Substitution Instances of Argument Forms

Just as there are substitution instances of propositional forms so there are substitution instances of argument forms. But whereas the substitution instances of propositions forms are propositions, the substitution instances of argument forms are arguments.

Once again, let us first represent the argument form

$$\begin{array}{c} (p \vee q) \rightarrow r \\ \sim r \\ \hline \sim p \bullet \sim q \end{array}$$

in the “box-notation” thus:

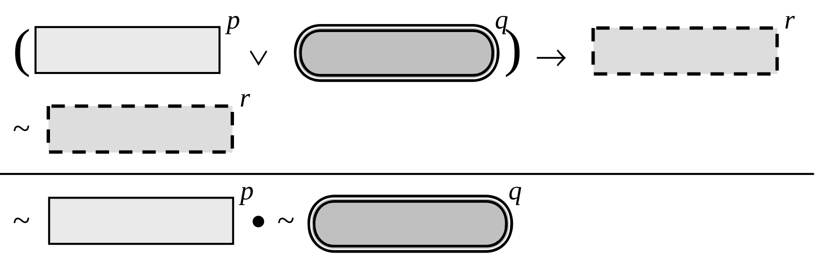


Consider three substitution keys again, each of which will generate a different substitution instance of this argument form

Key (A)

p := A
q := B
r := C

Given this substitution key, we produce the following substitution instance (fill in the variables!):



Thus the argument

$$\frac{\begin{array}{c}(A \vee B) \rightarrow C \\ \sim C\end{array}}{\sim A \bullet \sim B}$$

is a substitution instance (in fact, a proper substitution instance) of the argument form

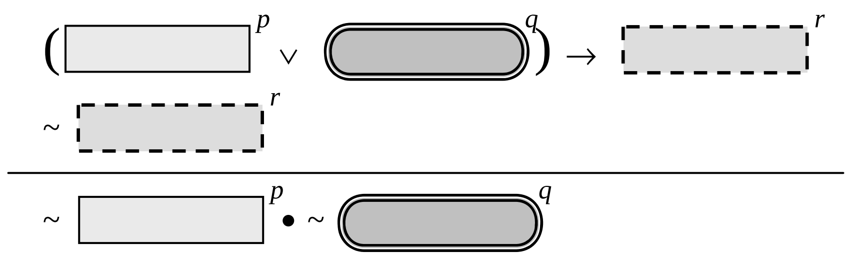
$$\frac{\begin{array}{c}(p \vee q) \rightarrow r \\ \sim r\end{array}}{\sim p \bullet \sim q}$$

Key (B)

As before, complex propositions can be substituted for propositional variables.

p := A • B
q := ~A ≡ C
r := ~~A

Given this substitution key, we produce the following substitution instance (fill in the variables!):



Note that when we insert A • B for p, we need to add parentheses around it – otherwise we would get an ambiguous formula, which would not be a proposition. We do not need to add parentheses around ~~A since we adopted the convention of dropping the parentheses around negations. Note that we *do* need to add parentheses around ~A ≡ C, which we will be substituting for q. ~A ≡ C is a biconditional, not a negation!

Thus the argument

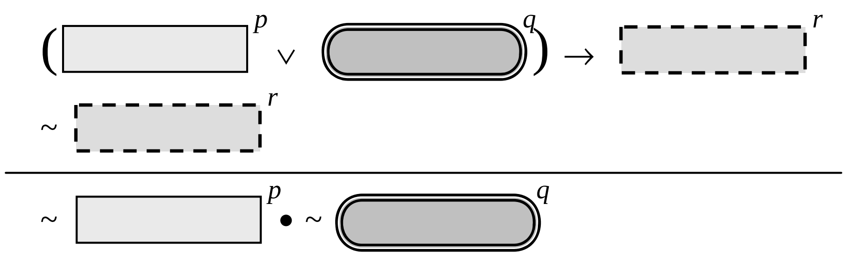
((A • B) ∨ (~A ≡ C)) → ~~A
~~~A
―――――――――――――
~(A • B) • ~(~A ≡ C)

is a substitution instance (though not a proper substitution instance) of the argument form

$(p \vee q) \rightarrow r$
$\sim r$
―――――――
$\sim p \bullet \sim q$
~~~

Key (C)

Again, the same proposition can be substituted for different variables.

p := A
q := A
r := A

Given this substitution key, we produce the following substitution instance (fill in the variables):



Thus the argument

$$\frac{\begin{array}{c}(\mathrm{A} \vee \mathrm{A}) \rightarrow \mathrm{A} \\ \sim\mathrm{A}\end{array}}{\sim\mathrm{A} \bullet \sim\mathrm{A}}$$

is a substitution instance (though not a proper substitution instance) of the argument form

$$\frac{\begin{array}{c}(p \vee q) \rightarrow r \\ \sim r\end{array}}{\sim p \bullet \sim q}$$

3.3. Substitution Exercises

In general, when you answer the substitution questions on the quiz, make sure (e.g. by writing on the side) that you can reconstruct a consistent substitution key. Consider the following exercise:

Mark all the substitution instances of the following argument form

p • (q → p)
p
so q

☐ 1. A • (B → A)
A
so B

☐ 2. A • ((A ∨ B) → A)
A
so B

☐ 3. C • ((A ∨ B) → A)
C
So A • B

☐ 4. (A ≡ C) • ((A ∨ B) → (A ≡ C))
A ≡ C
so A ∨ B

Let's consider all the cases in turn.

Case 1. Is

A • (B → A)
A
so B

a substitution instance of

p • (*q* → *p*)
p
so *q*

List the candidates for the substitution of *p*. In the first premise, *p* occurs as the first conjunct and as the consequent of the conditional, which is the second conjunct. So, in the first premise there is only one candidate substitution for *p*, viz. A. The only other occurrence of *p* in the argument is in the second premise, *p is* the second premise. Again there is only one candidate for the substitution for *p* in the second premise, viz. A. In the whole argument, A is the only one candidate for the substitution for *p*, so we can make a consistent substitution assignment for *p*.

What about *q*? *q* occurs twice in the argument. In the first premise, it occurs in the antecedent of the conditional, which is the second conjunct. There is only one candidate for the substitution of *q* in premise 1, viz. B. The second place *q* occurs is in the conclusion: *q* is the conclusion. So the second candidate for *q* is B. Again the same proposition. Therefore, there is a consistent way of substituting a proposition (viz. B) for *q*. Since we can construct a consistent substitution key for all the propositional variables:

q := B
p := A

The resulting argument (Case 1) form is a substitution instance (in fact, it is a proper substitution instance) of

$p \bullet (q \rightarrow p)$
p
so q

Case 2. Is

A • ((A ∨ B) → A)
A
so B

a substitution instance of

$p \bullet (q \rightarrow p)$
p
so q

List the candidates for the substitution of *p*. In the first premise, *p* occurs as the first conjunct and as the consequent of the conditional, which is the second conjunct. So, in the first premise there is only one candidate substitution for *p*, viz. A. The only other occurrence of *p* in the argument is in the second premise, *p is* the second premise. Again there is only one candidate for the substitution for *p* in the second premise, viz. A. In the whole argument, A is the only one candidate for the substitution for *p*, so we can make a consistent substitution assignment for *p*.

What about *q*? *q* occurs twice in the argument. In the first premise, it occurs in the antecedent of the conditional, which is the second conjunct. There is only one candidate for the substitution of *q* in premise 1, viz. A ∨ B. The second place *q* occurs is in the conclusion: *q* is the conclusion. So the second candidate for *q* is B. These are two different candidates for the substitution for *q*! Therefore, there is NO consistent way of substituting a proposition (there are two candidates B and A ∨ B) for *q*. The resulting argument (Case 2) is NOT a substitution instance of

$p \bullet (q \rightarrow p)$
p
so q

Case 3. Is

C • ((A ∨ B) → A)
C
A • B

a substitution instance of

$p \bullet (q \rightarrow p)$
p
so q

List the candidates for the substitution of *p*. In the first premise, *p* occurs as the first conjunct and as the consequent of the conditional, which is the second conjunct. So, in the first premise there are already two candidates for the substitution for *p*, viz. C (the first conjunct) and A (the consequent of the conditional, which is the second conjunct). This already suffices to establish that there is NO consistent way of substituting a proposition (there are two candidates C and A) for *p*, and hence that the resulting argument (Case 3) is NOT a substitution instance of

$p \bullet (q \rightarrow p)$
p
so q

Just for training, let's consider *q*. *q* occurs twice in the argument. In the first premise, it occurs in the antecedent of the conditional, which is the second conjunct. There is only one candidate for the substitution of *q* in premise 1, viz. A ∨ B. The second place *q* occurs is in the conclusion: *q* is the conclusion. So the second candidate for *q* is A • B. These are two different candidates for the substitution for *q*! Therefore, there is NO consistent way of substituting a proposition (there are two candidates A ∨ B and A • B) for *q*.

Finally,

Case 4. Is

(A ≡ C) • ((A ∨ B) → (A ≡ C))
A ≡ C
so A ∨ B

a substitution instance of

$p \bullet (q \rightarrow p)$
p
so q

List the candidates for the substitution of *p*. In the first premise, *p* occurs as the first conjunct and as the consequent of the conditional, which is the second conjunct. So, in the first premise there is only one candidate substitution for *p*, viz. A ≡ C. The only other occurrence of *p* in the argument is in the second premise, *p is* the second premise. Again there is only one candidate for the substitution for *p* in the second premise, viz. A ≡ C. In the whole argument, A ≡ C is the only one candidate for the substitution for *p*, so we can make a consistent substitution assignment for *p*.

What about *q*? *q* occurs twice in the argument. In the first premise, it occurs in the antecedent of the conditional, which is the second conjunct. There is only one candidate for the substitution of *q* in premise 1, viz. A ∨ B. The second place *q* occurs is in the conclusion: *q* is the conclusion. So the second candidate for *q* is A ∨ B. Again the same proposition. Therefore, there is a consistent way of substituting a proposition (viz. A ∨ B) for *q*. Since we can construct a consistent substitution key for all the propositional variables:

q := A ∨ B
p := A ≡ C

The resulting argument (Case 4) form is a substitution instance of

$p \bullet (q \rightarrow p)$
p
so q

Exercise "Substitution Instances" – 4

Construct the substitution instances of the listed propositional forms using the following substitution keys:

Key 1	**Key 2**
p := A	p := ~A
q := ~B	q := ~B
r := C ∨ D	r := ~B • C

Key 1 Key 2

(a)

$$\begin{array}{c} p \rightarrow q \\ \underline{p} \\ q \end{array}$$

(b)

$$\begin{array}{c} p \rightarrow q \\ \underline{q} \\ p \end{array}$$

(c)

$$\begin{array}{c} p \rightarrow q \\ \underline{\sim q} \\ \sim p \end{array}$$

Key 1	**Key 2**
p := A	p := ~A
q := ~B	q := ~B
r := C ∨ D	r := ~B • C

Key 1 Key 2

(d)

$$\frac{(p \bullet q) \to r \quad \sim r}{\sim p \lor \sim q}$$

______________	______________

(e)

$$\frac{\sim(p \equiv q) \lor r \quad \sim r \bullet \sim(p \to q)}{\sim(q \to p)}$$

______________	______________

(f)

$$\frac{(\sim p \to q) \equiv r \quad r \bullet \sim q}{p}$$

______________	______________

Exercise "Substitution Key" – 3

You are given an argument form and its substitution instance. Your task is to reconstruct the substitution key used to create the substitution instance:

(a)

$$\begin{array}{c} p \rightarrow q \\ {\sim}q \\ \hline {\sim}p \end{array} \qquad \begin{array}{c} A \rightarrow B \\ {\sim}B \\ \hline {\sim}A \end{array}$$

p :=

q :=

(b)

$$\begin{array}{c} p \vee {\sim}q \\ q \\ \hline p \end{array} \qquad \begin{array}{c} A \vee {\sim}A \\ A \\ \hline A \end{array}$$

p :=

q :=

(c)

$$\begin{array}{c} (p \equiv q) \vee p \\ p \bullet {\sim}q \\ \hline {\sim}p \end{array} \qquad \begin{array}{c} ({\sim}A \equiv {\sim}{\sim}C) \vee {\sim}A \\ {\sim}A \bullet {\sim}{\sim}{\sim}C \\ \hline {\sim}{\sim}A \end{array}$$

p :=

q :=

(d)

$$\begin{array}{c} (p \rightarrow q) \bullet q \\ p \vee {\sim}q \\ \hline {\sim}p \end{array} \qquad \begin{array}{c} ({\sim}C \rightarrow (A \vee B)) \bullet (A \vee B) \\ {\sim}C \vee {\sim}(A \vee B) \\ \hline {\sim}{\sim}C \end{array}$$

p :=

q :=

(e)

$$\begin{array}{c} (p \vee q) \bullet (q \vee p) \\ {\sim}q \\ \hline p \end{array} \qquad \begin{array}{c} ({\sim}B \vee A) \bullet (A \vee {\sim}B) \\ {\sim}A \\ \hline {\sim}B \end{array}$$

p :=

q :=

(f)

$$\begin{array}{c} (p \rightarrow q) \bullet (q \rightarrow p) \\ q \\ \hline p \end{array} \qquad \begin{array}{c} ({\sim}(B \bullet C) \rightarrow A) \bullet (A \rightarrow {\sim}(B \bullet C)) \\ A \\ \hline {\sim}(B \bullet C) \end{array}$$

p :=

q :=

(g)

$$\begin{array}{c} (p \rightarrow q) \bullet (q \rightarrow p) \\ {\sim}q \\ \hline {\sim}p \end{array} \qquad \begin{array}{c} ((C \vee D) \rightarrow B) \bullet (B \rightarrow (C \vee D)) \\ {\sim}B \\ \hline {\sim}(C \vee D) \end{array}$$

p :=

q :=

(h)

$$\begin{array}{c} (p \vee q) \bullet (q \rightarrow p) \\ {\sim}q \\ \hline p \end{array} \qquad \begin{array}{c} (A \vee ({\sim}B \equiv C)) \bullet (({\sim}B \equiv C) \rightarrow A) \\ {\sim}({\sim}B \equiv C) \\ \hline A \end{array}$$

p :=

q :=

Exercise "Substitution Instances" – 5

You are given an argument form in the center and nine potential substitution instances around it. Your task is to say which of the arguments are substitution instances of the argument form. For each of the substitution instances, reconstruct the substitution key used to create it:

(a)

1

$A \equiv B$
$\sim A$

$\sim B$

p :=
q :=

2

$A \equiv B$
$\sim B$

$\sim A$

p :=
q :=

3

$C \equiv B$
$\sim C$

$\sim B$

p :=
q :=

4

$\sim A \equiv \sim B$
$\sim A$

$\sim B$

p :=
q :=

$p \equiv q$
$\sim p$

$\sim q$

5

$\sim C \equiv B$
$\sim C$

$\sim B$

p :=
q :=

6

$(A \vee B) \equiv (C \bullet D)$
$\sim(A \vee B)$

$\sim(C \bullet D)$

p :=
q :=

7

$(A \vee B) \equiv (C \bullet D)$
$\sim A \vee B$

$\sim C \bullet D$

p :=
q :=

9

$\sim(A \vee B) \equiv (\sim C \bullet D)$
$\sim\sim(A \vee B)$

$\sim(C \bullet D)$

p :=
q :=

(b)

1

$A \rightarrow B$
$\sim A \vee \sim B$

$\sim(\sim A \bullet \sim B)$

p :=
q :=

2

$C \rightarrow D$
$\sim C \vee \sim D$

$\sim(C \bullet D)$

p :=
q :=

3

$A \rightarrow C$
$\sim(A \bullet C)$

$\sim A \vee \sim C$

p :=
q :=

4

$\sim A \rightarrow \sim B$
$\sim\sim A \vee \sim\sim B$

$\sim(\sim A \bullet \sim B)$

p :=
q :=

$p \rightarrow q$
$\sim p \vee \sim q$

$\sim(p \bullet q)$

5

$\sim A \rightarrow \sim A$
$\sim\sim A \vee \sim\sim A$

$\sim(\sim A \bullet \sim A)$

p :=
q :=

6

$(A \bullet B) \equiv \sim C$
$\sim(A \bullet B) \vee \sim\sim C$

$\sim((A \bullet B) \bullet \sim C)$

p :=
q :=

7

$(A \vee C) \rightarrow \sim(A \equiv B)$
$\sim(A \vee C) \vee \sim\sim(A \equiv B)$

$\sim((A \vee C) \bullet \sim(A \equiv B))$

p :=
q :=

9

$A \rightarrow B$
$A \vee B$

$\sim(\sim A \bullet \sim B)$

p :=
q :=

What You Need to Know and Do

- You need to know the distinction between a proposition and a propositional form in theory and in practice; you need to be able to tell when a proposition is a substitution instance, and when it is a proper substitution instance, of a propositional form;
- You need to know the distinction between an argument and an argument form in theory and in practice; you need to be able to tell when an argument is a substitution instance, and when it is a proper substitution instance, of an argument form.