Workbook Unit 4:

Calculating Truth-Values of Complex Propositions (Part I)

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Overview

In this unit and the next unit, we will learn to make use of the fact that all the connectives in propositional logic are truth-functional connectives, that is to say the truth-values of complex propositions is determined entirely by simple propositions. The technique of calculating truth-values of complex propositions is presupposed by a number of logical techniques developed in propositional logic, but it may also be useful in itself. You never know, after all, when you might encounter a teacher who would say: *You will get an A in this course if and only if the following proposition is true: "If it is both the case that London is the capital of the UK and that either Washington is the capital of the US but it is not the largest city in the US or Washington is no longer the capital of the US though it is the largest city in the US, then it is not true that London is the capital of Poland or the UK"*. It is sometimes useful to possess full-proof techniques for evaluating the truth-value of very complex propositions.

This unit

- teaches you to calculate the truth-values of arbitrarily complex propositions created by means of the two-place connectives.

PowerPoint Presentation

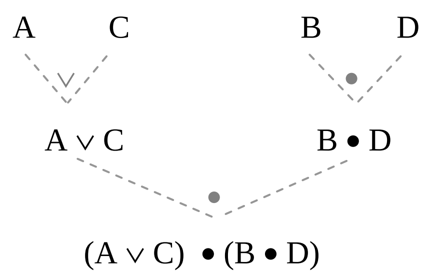
There is a PowerPoint Presentation that accompanies this Unit. It is available on-line as a .pps and a zipped .pps file.

1. Prerequisites

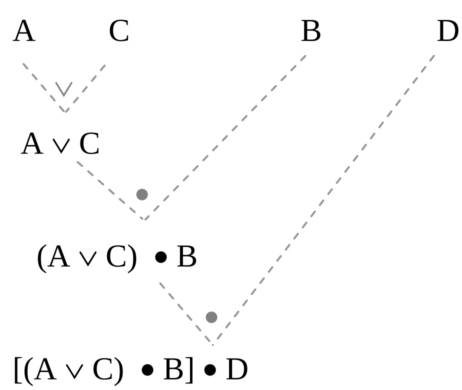
There are two prerequisites for this unit. First, you need to be able to determine the main connective for complex statements created by means of two-place connectives (§1.1 is a brief reminder, you should turn back to Unit 2 for more). Second, you need to learn the basic truth tables by heart.

1.1. Main Connective (Reminder)

Any given (well-formed) statement has only one main connective. When you think about how statements are constructed, the main connective is the last one that is put in. The construction of statement (A ∨ C) • (B • D) would be have the following stages:



Whereas the construction of [(A ∨ C) • B] • D would proceed somewhat differently:



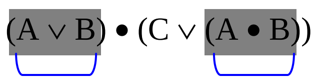
When statements are uncomplicated it is easy to determine what the main connective is. However, when statements are very complex, it is difficult to decide which connective is the main one. This is where the method of parentheses binding helps. It is a graphical method of binding parentheses pairwise so that the structure of the statement is revealed.

1.1.1. Example 1.

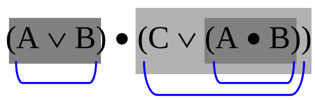
Consider the following statement.

$$(A \vee B) \bullet (C \vee (A \bullet B))$$

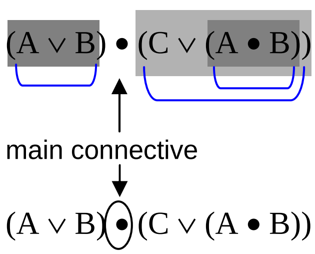
You might already know that the first dot is the main connective, that this is a conjunction of a disjunction, on the one hand, and a disjunction of C and a conjunction of A and B, on the other. But if this is difficult for you to see, you might proceed by first binding the innermost parentheses (those that bind statement letters) thus:



Once you have bound those, you bind the ones of the next level thus:



Now you have bound all parentheses. The structure of the statement should be evident.

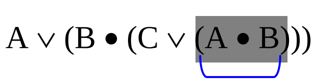


1.1.2. Example 2.

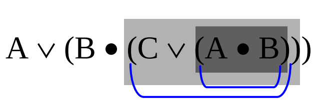
Consider the following statement.

$$A \vee (B \bullet (C \vee (A \bullet B)))$$

You might already know that the first wedge is the main connective, that this is a disjunction of a conjunction of a simple statement (B), on the one hand, and, on the other, a disjunction of C and a conjunction of A and B. But if this is difficult for you to see, you might proceed by first binding the innermost parentheses (those that bind statement letters) thus:



Once you have bound those, you bind the ones of the next level thus:



and the ones of the next level:

$$A \vee (B \bullet (C \vee (A \bullet B)))$$

Now you have bound all parentheses. The structure of the statement should be evident.

$$A \vee (B \bullet (C \vee (A \bullet B)))$$

main connective

$$A \vee (B \bullet (C \vee (A \bullet B)))$$

1.1.3. Example 3.

Consider the following statement.

$$(((A \vee B) \bullet C) \vee A) \bullet B$$

You might already know that the last dot is the main connective, that this is a conjunction, but you can see this more perspicuously thus:

$$(((A \vee B) \bullet C) \vee A) \bullet B$$

Once you have bound those, you bind the ones of the next level thus:

$$(((A \vee B) \bullet C) \vee A) \bullet B$$

and the ones of the next level:

$$(((A \vee B) \bullet C) \vee A) \bullet B$$

Now you have bound all parentheses. The structure of the statement should be evident.

$$(((A \vee B) \bullet C) \vee A) \bullet B$$

main connective

$$(((A \vee B) \bullet C) \vee A) \bullet B$$

1.2. Basic Truth Tables (Reminder)

You also need to know the basic truth tables by heart. Please, fill out the following truth tables and check that you have done so correctly. If not, make sure you go back to Unit 2 for further explanations.

p	*q*	*p* • *q*
T	T	
T	F	
F	T	
F	F	

p	*q*	*p* ∨ *q*
T	T	
T	F	
F	T	
F	F	

p	*q*	*p* ≡ *q*
T	T	
T	F	
F	T	
F	F	

p	*q*	*p* → *q*
T	T	
T	F	
F	T	
F	F	

2. Calculating the Truth-Values of Complex Propositions

Suppose I tell you that statements A and B are true while M is false. Since we are only working with truth-functional connectives, this means that we can determine the truth-value of no matter how complicated a statement constructed out of these statements, e.g. even ~[~(~A ∨ (B • M)) ≡ ~~~(A • (B → M))]! This might seem mind-boggling at first sight. So let's begin with simple cases.

Given our assumption, what is the truth-value of A ∨ B? (Note that you should know the answer immediately!) It is true. What about A • B? Likewise, it's true. How about A ∨ M? It's true. And A • M? Well, A • M is false because one of the conjuncts (M) is false.

OK. If we know this then we surely know what the truth-value of (A ∨ B) • (A • B) is. It's true because we already determined that A ∨ B and A • B are true (read the above paragraph, if you don't remember). What about: (A ∨ B) • (A • M)? It's false because the second conjunct (i.e. A • M) is, we said, false.

What about M ∨ [(A ∨ B) • (A • M)]? It is false. The first disjunct is false, and so is the second disjunct as we have calculated in the previous paragraphs.

And so on.

You get the general idea that the truth-value of compound statements depends on the truth-value of the components of those statements. In case those components are compound themselves, we need to look at their components and so on, until we reach simple statements whose truth-value is given to us.

Note! It is very important that you use your pen and paper and write out everything I say here. Your eyes and hand need to acquaint themselves with how this looks. For the first exercise, read it. Then read it again this time introducing all the steps in the area suggested. Then do the exercise again, this time on your own without looking. Check whether you have done it right.

2.1.1. A Method for Calculating Truth-Values

Suppose that A and B are true while M is false. Let's determine the truth-value of

(A ∨ M) • B

Basically what you do is to consider the structure of the statement but only with respect to whether the simple statements are true or false. So you would write something that looks thus:

```
(T ∨ F) • T
```

(Note that I'm using a different font since strictly speaking this is not a statement at all.) What you do is work from within (in the order that reflects the structure of the statement, as before). This statement is a conjunction but we do not know the truth-value of the first conjunct, which is a compound statement – a disjunction. We don't know the truth-value of the disjunction either. Before we can calculate it – the disjunction of a true and a false statement is true, hence we replace `(T ∨ F)` with `(T)`:

```
(T ∨ F) • T
  (T)   • T
```

Some like to leave the parentheses around `(T)` as a reminder that this truth-value is the result of calculating the truth-value of a parenthesized compound statement, but the above is just equivalent to:

```
(T ∨ F) • T
   T    • T
```

(Note that the truth-value of the second conjunct is just copied in the process. We cannot do anything with it until we know the truth-value of the first conjunct.)

And now what remains is the calculation of the truth-value of the conjunction, which is true because both conjuncts are true, replacing `T • T` with `T`:

```
   T    • T

        T
```

This is what the calculation tree would look like without my comments:

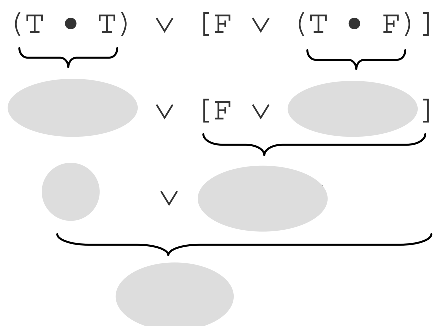
```
(T ∨ F) • T
  (T)   • T
        T
```

Example 1

Let us do another example together. Try to do this on your own first. I will be providing commentary further. Suppose A and B are true and M is false.

$$(A \bullet B) \vee [M \vee (A \bullet M)]$$

We first need to substitute the truth-values remembering to leave in all the parentheses (they tell you what to do first):



We start working from within. The conjunction (T • T) is going to be true, while the conjunction (T • F) is going to be false.

```
(T • T) ∨ [F ∨ (T • F)]
  (T)   ∨ [F ∨   (F)  ]
```

In the next line, we can only copy the truth-value for the first disjunct of the main disjunction, but we can calculate the disjunction in the brackets:

```
  (T)   ∨ [F ∨   (F)  ]
   T    ∨   [F]
```

Finally, in the last the disjunction is calculated. Without commentary, your calculation ought to look thus:

```
(T • T) ∨ [F ∨ (T • F)]
  (T)   ∨ [F ∨   (F)]
   T    ∨   [F]
        T
```

Example 2

Suppose A, B and C are true and M is false. What is the truth-value of:

$$A \equiv (B \rightarrow (C \bullet M))$$

After the substitution of the truth-values, we can proceed with the calculation:

```
T ≡ (T → (T • F))
T ≡ (T →         )
T ≡
```

Here the calculation is quite straightforward. We cannot calculate anything until we know the truth-values in the innermost parentheses. So this is where we start:

```
T ≡ (T → (T • F))
T ≡ (T →   (F)   )
```

Once we know the truth-value of the consequent, we can calculate the truth-value of the conditional:

```
T ≡ (T →     F    )
T ≡   (F)
```

Once we know the truth-value of the second term of the biconditional, we can calculate the truth-value of the biconditional – it is false:

```
T ≡   (F)
  F
```

The calculation will look thus:

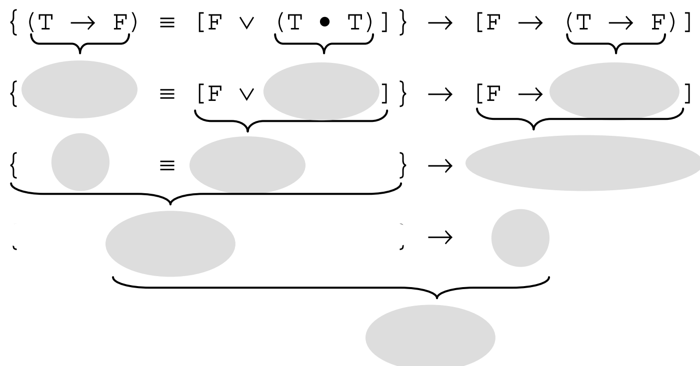
```
T ≡ (T → (T • F))
T ≡ (T →   (F))
T ≡   (F)
  F
```

Example 3

Suppose again that A and B are true and M and N are false. What is the truth-value of:

$$\{(A \rightarrow M) \equiv [N \vee (A \bullet B)]\} \rightarrow [M \rightarrow (A \rightarrow N)]$$

We first need to substitute the truth-values remembering to leave in all the parentheses:



Again, we starting working from within. When we resolve the inner parentheses, we simplify our task. Let's do so progressively:

```
{ (T → F) ≡ [F ∨ (T • T)] } → [F → (T → F)]
{   (F)   ≡ [F ∨   (T)  ] } → [F →   (F)  ]
```

This last line becomes in effect (after dropping the spurious parentheses around single truth-values):

```
{F ≡ [F ∨ T]} → [F → F]
```

Again we should work on the innermost parentheses:

```
{F ≡ [F ∨ T]} → [F → F]
{F ≡   [T]  } →   [T]
```

Now we can calculate the truth-value of the antecedent of the main conditional:

```
{F ≡  [T] } →  [T]
  {F}       →   T
```

The only thing that remains is the calculation of the truth-value of the conditional – it will be true since its antecedent is false. Without the commentary:

```
{ (T → F) ≡ [F ∨ (T • T)] } → [F → (T → F)]
{   (F)   ≡ [F ∨   (T)  ] } → [F →   (F)  ]
{   (F)   ≡    [T]        } →      [T]
          {F}               →      [T]
                     T
```

Exercise Truth-Values (A) – 1

In this exercise, you will be provided with the schemata for carrying out the calculations. Make sure that you understand why you proceed in this order. In the next exercises, you will be "on your own." Check that your results match the results in the Solutions before proceeding.

1. (T • T) ∨ (F → F)

   ∨

2. T • (T ∨ (F → F))

   T • (T ∨)

   T •

3. ((T • T) ∨ F) → F

   (∨ F) → F

   → F

4. T • ((T ∨ F) → F)

   T • (→ F)

   T •

5. (T → F) • ((T • F) → F)

   • (→ F)

   •

6. (T ≡ (T ∨ F)) → F

   (T ≡) → F

   → F

7. (F → T) → (T → F)

   →

8. F → (F → (F → F))

   F → (F →)

   F →

9. ((F → F) → F) → F

   (→ F) → F

   → F

10. ((F → T) → F) ≡ (F ≡ F)

   ( ) → F) ≡

   ≡

Exercise Truth-Values (A) – 2

Carry out the following calculations step-by-step, rewriting all the necessary connectives and parentheses. Do not make shortcuts. Check that your results match the results in the Solutions before proceeding.

1. (F • T) ∨ (F → F)

2. F • (T ∨ (F → F))

3. ((F • T) ∨ F) → F

4. F • ((T ∨ F) → F)

5. (T → F) → ((T • F) → F)

6. (F • (T ∨ F)) → F

7. (F → F) → (F → T)

8. F → (F → (F → T))

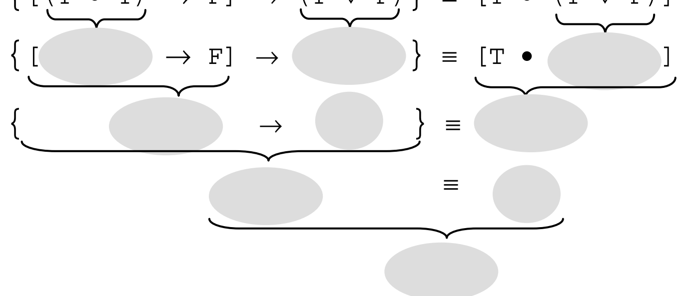
9. ((F → T) → F) → T

10. ((F ∨ T) → F) → (F → F)

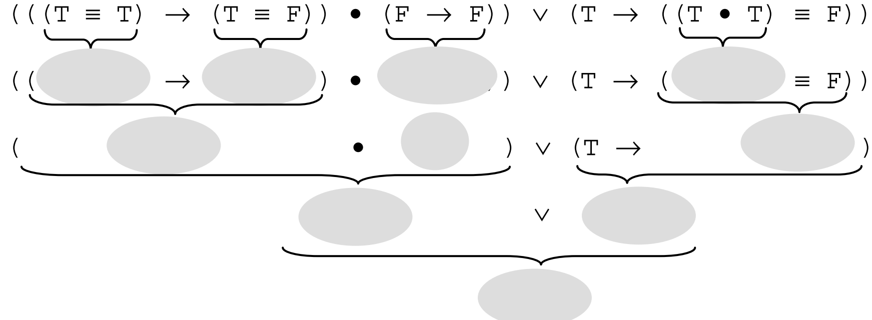
Exercise Truth-Values (A) – 3

In this exercise, you will be provided with the schemata for carrying out the calculations. Make sure that you understand why you proceed in this order. In the next exercises, you will be "on your own." Check that your results match the results in the Solutions before proceeding.

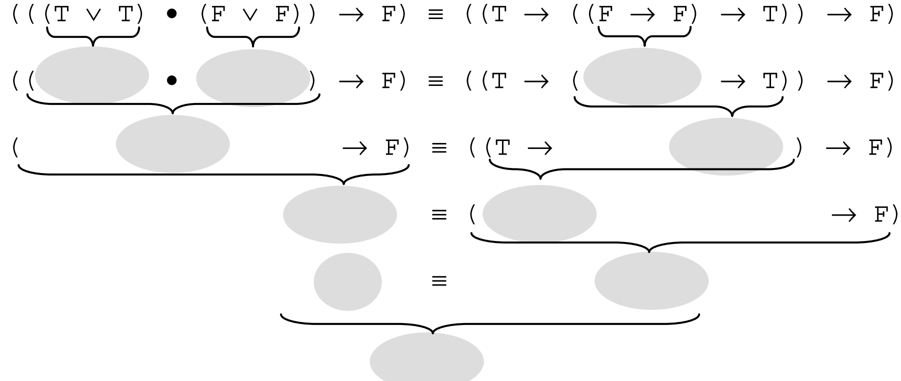
1. {[(T • T) → F] → (T ∨ F)} ≡ [T • (T ∨ F)]



2. (((T ≡ T) → (T ≡ F)) • (F → F)) ∨ (T → ((T • T) ≡ F))



3. (((T ∨ T) • (F ∨ F)) → F) ≡ ((T → ((F → F) → T)) → F)



Exercise Truth-Values (A) – 4

Carry out the following calculations step-by-step, rewriting all the necessary connectives and parentheses. Do not make shortcuts. Check that your results match the results in the Solutions before proceeding.

1. [((T • F) • F) ≡ (T ∨ F)] → (T • (F ∨ F))

2. ((T • F) • F) ≡ [(T ∨ F) → (T • (F ∨ F))]

3. ((T ≡ F) → (F ≡ F)) • [(F → F) ∨ (F → ((T • F) ≡ F))]

4. (T ∨ F) • {[(F ∨ F) → F] ≡ [T → [((F → F) → F) → F]]}

Exercise Truth-Values (A) – 5

On the assumption that A, B and C are true while K, L and M are false, what is the truth-value of the following propositions:

1. $(A \rightarrow M) \equiv [L \vee (B \bullet K)]$

2. $A \rightarrow [L \equiv (C \vee (K \bullet B))]$

3. $(((K \rightarrow B) \equiv L) \vee M) \bullet B$

4. $((K \rightarrow M) \equiv C) \vee (A \bullet M)$

5. $[((L \vee A) \vee M) \vee B] \equiv [(L \bullet A) \bullet (M \bullet B)]$

6. $[((K \rightarrow B) \equiv L) \rightarrow M] \equiv [((M \rightarrow C) \rightarrow M) \rightarrow M]$

3. Summary

In this unit, you have acquired the skill of calculating the truth-value of even very complicated propositions (created by means of two-place connectives) given the truth-values of simple propositions.

What You Need to Know and Do

- You need be able to calculate the truth-value of even very complicated propositions using the step-by-step method introduced in this unit..